Heat Equation: Explicit Formulas

(First 25 minutes: Quiz 1)

We now turn to the heat equation

$$u_t - \triangle u = f, \quad \text{in } U_T; \qquad u = g \quad \text{on } \partial^* U_T \tag{1}$$

where

$$U_T \equiv U \times [0, T); \qquad \partial^* U_T \equiv \left(\bar{U} \times \{0\}\right) \cup (\partial U \times [0, T]). \tag{2}$$

with $U \subset \mathbb{R}^n$.

We call $\partial^* U_T$ the reduced boundary of U_T. In this lecture we will find explicit representation formula via fundamental solution, and discuss its maximum principles.

1. Fundamental solutions and homogeneous initial-value problems.

1.1. Deriving the fundamental solution.

Similar to the case of Laplace/Poisson equations, we seek a special solution in the case $\Omega = \mathbb{R}^n$ which can help representing other solutions. We do this through Fourier transform.

We consider the Fourier transform in the spatial variable for the initial value problem

$$u_t - \triangle u = 0, \quad t > 0; \qquad u = g, \quad t = 0. \tag{3}$$

We obtain an ODE for the function $\hat{u}(\xi, t)$:

$$(\hat{u})_t + |\xi|^2 \hat{u} = 0, \qquad \hat{u}(\xi, 0) = \hat{g}(\xi). \tag{4}$$

This equation is easy to solve:

$$\hat{u}(\xi, t) = \hat{g}(\xi)\, e^{-|\xi|^2 t}. \tag{5}$$

Due to the following property of the Fourier transform:

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\, \hat{g}(\xi)$$

Thus all we need to do is to find the inverse Fourier transform of $e^{-|\xi|^2 t}$. From properties of the Fourier transform, if

$$\hat{G}(\xi) = e^{-|\xi|^2}, \tag{6}$$

then the Fourier transform of $t^{-n/2} G\left(x/t^{1/2}\right)$ is $e^{-|\xi|^2 t}$.

Lemma 1. *Let* $G(x) = \frac{1}{(4\pi)^{-n/2}} e^{-|x|^2/4}$, *then* $\hat{G}(\xi) = e^{-|\xi|^2}$.

Proof. All we need to do is to compute

$$\frac{1}{(4\pi)^{-n/2}} \int_{\mathbb{R}^n} e^{-|x|^2/4} e^{-i x \cdot \xi}\, \mathrm{d}x. \tag{7}$$

We have

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-|x|^2/4} e^{-i x \cdot \xi}\, \mathrm{d}x &= \int_{\mathbb{R}^n} e^{-\frac{x_1^2}{4} - \frac{x_2^2}{4} - \cdots - \frac{x_n^2}{4} - i x_1 \xi_1 - i x_2 \xi_2 - \cdots - i x_n \xi_n}\, \mathrm{d}x \\ &= \left(\int_{\mathbb{R}} e^{-\frac{x_1^2}{4} - i x_1 \xi_1}\, \mathrm{d}x_1 \right) \cdots \left(\int_{\mathbb{R}} e^{-\frac{x_n^2}{4} - i x_n \xi_n}\, \mathrm{d}x_n \right). \end{aligned} \tag{8}$$

Thus our task is to compute the integral

$$\int_{\mathbb{R}} e^{-\frac{x^2}{4} - i x \xi}\, \mathrm{d}x \tag{9}$$

where both x and ξ are scalars now.

We construct the following contour in $\mathbb{C}$: Let $R > 0$ be real.

$$\{-R \to +R\} \cup \{R \to R - 2\xi i\} \cup \{R - 2\xi i \to -R - 2\xi i\} \cup \{-R - 2\xi i \to -R\} \equiv \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4. \tag{10}$$

The function

$$e^{-z^2/4 - i z \xi} \tag{11}$$

is analytic inside this contour, therefore

$$\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} = 0. \tag{12}$$

We further notice that

$$\left| \int_{\Gamma_2(\Gamma_4)} e^{-z^2/4 - i z \xi} \right| \leqslant C e^{-R^2/4} \rightarrow 0 \tag{13}$$

and

$$\int_{\Gamma_1} e^{-z^2/4 - i z \xi} \rightarrow \int_{\mathbb{R}} e^{-\frac{x^2}{4} - i x \xi} \, \mathrm{d}x \tag{14}$$

as $R \nearrow +\infty$. Thus

$$\int_{\mathbb{R}} e^{-\frac{x^2}{4} - i x \xi} \, \mathrm{d}x = \lim_{R \nearrow \infty} - \int_{\Gamma_3} e^{-z^2/4 - i z \xi} \, \mathrm{d}z. \tag{15}$$

We compute

$$\begin{aligned} - \int_{\Gamma_3} e^{-z^2/4 - i z \xi} \, \mathrm{d}z &= \int_{-R-2\xi i}^{R-2\xi i} e^{-z^2/4 - i z \xi} \, \mathrm{d}z \\ &= \int_{-R}^{R} e^{-(y-2\xi i)^2/4 - i(y-2\xi i)\xi} \, \mathrm{d}y \\ &= \int_{-R}^{R} e^{-y^2/4} e^{-\xi^2} \, \mathrm{d}y \rightarrow (4\pi)^{1/2} e^{-\xi^2}. \end{aligned} \tag{16}$$

□

Definition 2. *The function*

$$\Phi(x,t) \equiv \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t = 0 \end{cases} \tag{17}$$

is called the fundamental solution of the heat equation.

1.2. Properties of the fundamental solution.

The fundamental solution enjoys the following properties.

1. For each time $t > 0$,

$$\int_{\mathbb{R}^n} \Phi(x,t) \, \mathrm{d}x = 1. \tag{18}$$

2. Treating t as a parameter,

$$\lim_{t \searrow 0} \Phi(x,t) = \delta \tag{19}$$

in the sense of distributions. In other words, we have

$$\lim_{t \searrow 0} \int_{\mathbb{R}^n} \Phi(x,t) \, f(x) = f(0) \tag{20}$$

for any continuous function f.

1.3. Homogeneous initial value problem.

From the above properties we immediately obtain the explicit formula for solutions for the initial-value problem:

$$u_t - \triangle u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty); \qquad u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \tag{21}$$

Theorem 3. (Solution of initial-value problem) *Assume* $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *and define* u *by*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) \, g(y) \, \mathrm{d}y = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) \, \mathrm{d}y, \qquad t > 0, \tag{22}$$

then

i. $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$,

ii. $u_t - \triangle u = 0$ *when* $t > 0$,

iii. u *takes* g *as its initial value, that is*

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x\in\mathbb{R}^n, t>0}} u(x,t) = g(x_0) \tag{23}$$

for all $x_0 \in \mathbb{R}^n$.

Proof. See Evans P. 47. □

2. Nonhomogeneous problem, Duhamel's principle.

Now let us consider the nonhomogeneous case

$$u_t - \triangle u = f \quad \text{in } \mathbb{R}^n \times (0,\infty); \qquad u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \tag{24}$$

It is clear that we can immediately simplify the situation to the case of zero initial data:

$$u_t - \triangle u = f \quad \text{in } \mathbb{R}^n \times (0,\infty); \qquad u = 0 \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \tag{25}$$

By *Duhamel's principle* we can write down the solution:

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\, f(y,s)\, \mathrm{d}y\, \mathrm{d}s = \int_0^t \frac{1}{(4\pi (t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, \mathrm{d}y\, \mathrm{d}s. \tag{26}$$

Now we can prove the following theorem.

Theorem 4. (Solution of nonhomogeneous problem) *Let* $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$[1] *and have compact support. Then*

i. $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$,

ii. $u_t - \triangle u = f$ *for* $t > 0$,

iii. *For each* $x_0 \in \mathbb{R}^n$,

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x\in\mathbb{R}^n, t>0}} u(x,t) = 0. \tag{27}$$

Proof. See Evans p. 50. □

Combining the above results, we can present the formula for the solution in the general case in the whole space:

$$u_t - \triangle u = f \quad t > 0; \qquad u = g \quad t = 0. \tag{28}$$

The solution is

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, \mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\, f(y,s)\, \mathrm{d}y\, \mathrm{d}s. \tag{29}$$

3. Regularity.

Theorem 5. (Smoothing effect of heat kernel) *Suppose* $u \in C_1^2(\bar{\Omega}_T)$ *solves the heat equation in* Ω_T, *then*

$$u \in C^\infty(\Omega_T). \tag{30}$$

We introduce the typical region considered when doing parabolic regularity:

$$C(x,t;r) \equiv \{(y,s):\ |x-y| \leqslant r,\ t - r^2 \leqslant s \leqslant t\}. \tag{31}$$

1. $C_n^m(\mathbb{R}^n \times [0,\infty))$ *means* f *has* m *continuous derivatives in* x *and* n *continuous derivatives in* t.

It suffices to prove that if $u \in C_1^2(C(x,t;r))$ solves the equation, then $u \in C^\infty$ inside $C(x,t;r/2)$.

Proof. See Evans pp. 59 – 61. The main idea is the following.

Fix (x_0, t_0). Take a "cut-off" function η which is 0 outside $C(x_0, t_0; r)$ and 1 inside $C(x_0, t_0; 3\,r/4)$. Then consider $v(x,t) \equiv \eta(x,t)\, u(x,t)$. We have

$$v_t - \triangle v = \eta_t\, u - 2\, \nabla \eta \cdot \nabla u - u\, \triangle \eta. \tag{32}$$

Now we can use the explicit formula to obtain

$$u(x,t) = v(x,t) = \int_{C(x_0,t_0;r)} \Phi(x-y, t-s)\, [(\eta_s - \triangle \eta)\, u - 2\, \nabla \eta \cdot \nabla u](y,s)\, \mathrm{d}y\, \mathrm{d}s. \tag{33}$$

for all $(x,t) \in C(x_0, t_0; 3\, r/4)$.

Finally notice that, $\eta_s, \triangle \eta, \nabla \eta$ vanishes inside $C(x_0, t_0; 3\, r/4)$ which means

$$u(x,t) = \int_{C(x_0,t_0;r) \backslash C(x_0,t_0;3r/4)} \Phi(x-y, t-s)\, [(\eta_s - \triangle \eta)\, u - 2\, \nabla \eta \cdot \nabla u](y,s)\, \mathrm{d}y\, \mathrm{d}s. \tag{34}$$

As a consequence, for any $(x,t) \in C(x_0, t_0; r/2)$, the integrand is uniformly bounded and can be differentiated arbitrarily. □

We further have the following estimate for derivatives.

Theorem 6. *We have*

$$\max_{C(x,t;r/2)} \left| \partial_x^\alpha \partial_t^l u \right| \leqslant \frac{C(\alpha, l)}{r^{k+2l+n+2}} \int_{C(x,t;r)} |u|\, \mathrm{d}x\, \mathrm{d}t. \tag{35}$$

Proof. This follows naturally from the proof of the last theorem. Evans pp. 61 – 62 for details. □