PROPERTIES AND ESTIMATES OF LAPLACE'S AND POISSON'S EQUATIONS

In our last lecture we derived the formulas for the solutions of Poisson's equation through Green's function:

$$u(x) = \int_U G(x,y)\, f(y)\, \mathrm{d}y - \int_{\partial U} \frac{\partial G(x,y)}{\partial n_y}\, g(y)\, \mathrm{d}S_y \tag{1}$$

solves

$$-\triangle u = f \quad \text{in } U; \qquad u = g \quad \text{on } \partial U. \tag{2}$$

(When $U = \mathbb{R}^n$, the boundary term vanishes, and $G(x,y)$ should be replaced by the fundamental solution $\Phi(x-y)$).

From this formula we can obtain many regularity estimates for u, see the lecture note "Hölder Estimates for the Poisson Equation" for Fall 2008 Math527, or the GTM book "Partial Differential Equations" by J. Jost for more details.

Note that, such estimates only apply to u given by the above formula. In other words, before we settle the uniqueness issue, we cannot use the above formula to estimate general solutions of Poisson's equation.

Naturally, the uniqueness question leads to the study of Laplace's equation with zero boundary condition:

$$\triangle u = 0 \quad \text{in } U; \qquad u = 0 \quad \text{on } \partial U. \tag{3}$$

Thus it is important to study the properties of C^2 functions satisfying $\triangle u = 0$. Such functions are called *harmonic functions*.

Remark 1. We will see soon that, somehow surprisingly, the study of harmonic functions leads to much more than uniqueness of Poisson's equation. It turns out that all estimates can be obtained through several properties of the equation $\triangle u = 0$ and the related $\triangle u \geqslant (\leqslant) 0$, without using the exact formula above.

Remark 2. Another reason of studying harmonic functions is that the properties are much more stable under perturbation of the equation itself. While the exact formula only applies to Poisson equation, the properties of harmonic functions are shared by general linear elliptic equation

$$\nabla \cdot (A(x) \cdot Du) = f \tag{4}$$

and even nonlinear equations.

1. Properties of harmonic functions.

Recall the definition

Definition 3. *A C^2 function satisfying $\triangle u = 0$ in U is called a harmonic function in U.*

1.1. Mean value formula.

Theorem 4. *If $u \in C^2(U)$ is harmonic, then*

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u\, \mathrm{d}x = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u\, \mathrm{d}S \tag{5}$$

for every ball $B_r(x) := \{y \mid |y - x| < r\} \Subset U$.

Remark 5. It turns out that

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u\, \mathrm{d}S \tag{6}$$

is easier to prove. Thus we need to first establish

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u\, \mathrm{d}x \text{ for all } B_r(x) \Subset \Longleftrightarrow u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u\, \mathrm{d}S \text{ for all } B_r(x) \Subset U. \tag{7}$$

This is left as an exercise.

Proof. We prove

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \, \mathrm{d}S \text{ for all } B_r(x) \Subset U.$$

Without loss of generality, set $x = 0$ and denote $B_r(0)$ by B_r.

We compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{|\partial B_r|} \int_{\partial B_r} u \, \mathrm{d}S\right] &= \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{|\partial B_1|} \int_{\partial B_1} u(r\, w) \, \mathrm{d}S_w\right] \\ &= \frac{1}{|\partial B_1|} \int_{\partial B_1} w \cdot Du(r\, w) \, \mathrm{d}S_w \\ &= \frac{1}{|\partial B_1|} \int_{\partial B_1} \boldsymbol{n} \cdot Du(y) \, \mathrm{d}S_y \\ &= \frac{1}{|\partial B_1|} \int_{B_1} \triangle u \, \mathrm{d}y = 0. \end{aligned} \tag{8}$$

Thus

$$\frac{1}{|\partial B_r|} \int_{\partial B_r} u \, \mathrm{d}S = \lim_{r \searrow 0} \frac{1}{|\partial B_r|} \int_{\partial B_r} u \, \mathrm{d}S = u(0) \tag{9}$$

due to the continuity of u. □

Theorem 6. *If $u \in C^2(U)$ satisfies*

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \, \mathrm{d}S \tag{10}$$

or

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u \, \mathrm{d}x \tag{11}$$

for all $x \in U$ and all balls $B_r(x) := \{y \mid |y - x| < r\} \Subset U$, then u is harmonic.

Proof. We have already seen that the two conditions are equivalent. Thus we only need to show that

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \, \mathrm{d}S \text{ for all } x \in U, B_r(x) \Subset U \implies \triangle u = 0 \text{ in } U. \tag{12}$$

This can be fulfilled by simply reverse the argument in the proof of the above theorem. □

Remark 7. The above "Converse to mean-value property" is kind of trivial and not very useful. If u is already C^2, we can simply differentiate to see whether at every x $\triangle u = 0$ or not, and there is no need to check the mean value condition for every x and every ball.

What makes the mean value formula useful is the following theorem, which says we do not need the a priori knowledge that u is C^2.

Theorem 8. *If $u \in C(U)$ satisfies*

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \, \mathrm{d}S \tag{13}$$

for all $x \in U$ and all balls $B_r(x) := \{y \mid |y - x| < r\} \Subset U$, then u is harmonic.

Proof. Since we already have shown that the mean value property leads to u harmonic if $u \in C^2$, we only need to show $u \in C^2$.

Take any radially symmetric function $\phi = \phi(r)$ supported in B_ε with $\int_{B_\varepsilon} \phi = 1$. We will show that

$$(u * \phi)(x) = \int_{\mathbb{R}^n} u(y) \, \phi(x - y) \, \mathrm{d}y = u(x). \tag{14}$$

Now recall that

$$D^\alpha(u * \phi) = u * (D^\alpha \phi). \tag{15}$$

Obviously we can take $\phi \in C^2$ and conclude that $u \in C^2$.

Now we show

$$(u * \phi)(x) = \int_{\mathbb{R}^n} u(y)\, \phi(x - y)\, \mathrm{d}y = u(x). \tag{16}$$

Without loss of generality, set $x = 0$. Take ε so small such that $B_\varepsilon \Subset U$. We compute

$$\begin{aligned} (u * \phi)(0) &= \int_{\mathbb{R}^n} u(y)\, \phi(-y)\, \mathrm{d}y \\ &= \int_{B_\varepsilon} u(y)\, \phi(-y)\, \mathrm{d}y \\ &= \int_0^\varepsilon \left[\int_{\partial B_r} u(y)\, \mathrm{d}S_y \right] \phi(r)\, \mathrm{d}r \\ &= \int_0^\varepsilon |\partial B_r|\, u(0)\, \phi(r)\, \mathrm{d}r \\ &= u(0) \left[\int_0^\varepsilon \int_{\partial B_r} \phi(r)\, \mathrm{d}S\, \mathrm{d}r \right] \\ &= u(0) \int_{B_\varepsilon} \phi(y)\, \mathrm{d}y \\ &= u(0). \end{aligned} \tag{17}$$

Thus ends the proof. □

Corollary 9. *If u is harmonic, then $u \in C^\infty$.*

Proof. As u is harmonic, u satisfies the mean value formula. Therefore

$$u * \phi = u \tag{18}$$

for all ϕ satisfying the condition in the above theorem. Taking $\phi \in C^\infty$ gives the conclusion. □

The same argument can in fact prove the following Weyl's lemma, which relaxes the condition $u \in C(U)$. Those who are interested in its proof can take a look at the lecture note "Harmonic Functions" for Fall 2008's Math 527.

Lemma 10. (Weyl's lemma) *Let $u: U \mapsto \mathbb{R}$ be measurable and locally integrable in Ω. Suppose that for all $\varphi \in C_0^\infty(U)$,*

$$\int_U u(x)\, \triangle\varphi(x)\, \mathrm{d}x = 0. \tag{19}$$

Then u is harmonic and, in particular, smooth.

Remark 11. A question for those who know what a distribution:

Let u be a distribution and

$$\triangle u = 0 \tag{20}$$

in the distributional sense. Then can we conclude that u is C^∞?

1.2. Local estimates for harmonic functions.

Using the mean value formula, we can obtain good estimates for the derivatives of hamonic functions (recall that harmonic functions are C^∞).

Theorem 12. *Assume u is harmonic in U. Then*

$$|D^\alpha u(x)| \leqslant \frac{C_k}{r^{n+k}} \int_{B_r(x)} |u|\, \mathrm{d}x \tag{21}$$

for each $x \in U$ and $B_r(x) \Subset U$.

Proof. See p. 29 of Evans. □

Remark 13. From the above estimates, it is easy to show that u is not only C^∞, but in fact analytic. See pp. 31 – 32 of Evans.

Remark 14. The mean value formulas cease to be true for Poisson's equation or the more general elliptic equations. As a consequence, one can not obtain local estimates for these equations using the above method. A more robust way is to estimate through the following maximum principles.

1.3. Harnack inequality.

It turns out that, for nonnegative harmonic functions, its value at two different points are always comparable.

Theorem 15. (Harnack's inequality) *For each connected open set $V \Subset U$, there exists a positive constant C, depending only on V, such that*

$$\sup_V u \leqslant C \inf_V u \tag{22}$$

for all nonnegative harmonic functions u in U.

Proof. First consider two points x, y (denote $r := \operatorname{dist}(x, y)$), such that $B_{2r}(x) \Subset U$. Using mean value formula we have

$$u(x) = \frac{1}{|B_{2r}|} \int_{B_{2r}(x)} u \geqslant \frac{1}{|B_{2r}|} \int_{B_r(y)} u = \frac{1}{2^n} u(y). \tag{23}$$

The conclusion easily follows. □

1.4. Uniqueness for Poisson equation.

It suffices to establish the following maximum principle:

Theorem 16. (Weak maximum principle) *Suppose $u \in C^2(U) \cap C(\bar{U})$ is harmonic in U. Then*

$$\max_{\bar{U}} u = \max_{\partial U} u. \tag{24}$$

Proof. Assume the contrary, that is $\max_{\bar{U}} u > \max_{\partial U} u$. Then there must be a x_0 such that

$$u(x_0) = \max_U u \tag{25}$$

but $u \not\equiv u(x_0)$ in some neighborhood of x_0. This contradicts the mean value formula. □

In fact, one can establish the stronger

Theorem 17. (Strong maximum principle) *Suppose $u \in C^2(U) \cap C(\bar{U})$. If U is connected and there exists a point $x_0 \in U$ such that*

$$u(x_0) = \max_{\bar{U}} u, \tag{26}$$

then $u \equiv u(x_0)$ in U.

Proof. See p. 27 of Evans. □

Remark 18. It is clear that the strong maximum principle ceases to be true when U is not connected.

Theorem 19. *The solution to Poisson's equation is unique.*

Proof. It follows from applying the weak maximum principle to the equation

$$\triangle u = 0 \tag{27}$$

with 0 boundary condition. □

2. Maximum principles.

2.1. Subharmonic and superharmonic functions.

We consider, instead of $\triangle u = 0$, the inequalities

$$-\triangle u \leqslant (\geqslant) 0. \tag{28}$$

A simple adaptation of the proof for the Laplace equation then gives

$$-\triangle u \leqslant (\geqslant) 0 \Longrightarrow u(x) \leqslant (\geqslant) \frac{1}{|B_r(x)|} \int_{B_r(x)} u \,\mathrm{d}x, \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \,\mathrm{d}S \tag{29}$$

for all $x \in U$, $B_r(x) \Subset U$. This naturally leads to the following definition.

Definition 20. *Let u be continuous. It is called subharmonic (superharmonic) if for every $B_r(x) \Subset U$, we have*

$$u(x) \leqslant (\geqslant) \frac{1}{|B_r(x)|} \int_{B_r(x)} u \,\mathrm{d}x \text{ or } \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \,\mathrm{d}S. \tag{30}$$

Remark 21. It is easy to see that subharmonic/superharmonic functions are not necessarily differentiable, as the 1D example $u = 1 - |x|$ shows.

Remark 22. One can show that, v is subharmonic(superharmonic) if and only if for every $V \Subset U$, and every harmonic function u on V such that $u \geqslant (\leqslant) v$ on ∂V, we have

$$u \geqslant (\leqslant) v \text{ in } V. \tag{31}$$

This further justifies the terminology "subharmonic" ("superharmonic").

Remark 23. Question:

Do we still have

$$u(x) \leqslant (\geqslant) \frac{1}{|B_r(x)|} \int_{B_r(x)} u \,\mathrm{d}x \Longleftrightarrow u(x) \leqslant (\geqslant) \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \,\mathrm{d}S \tag{32}$$

or not?

It is easy to show that

- u subharmonic, then
$$\max_{\bar{U}} u \leqslant \max_{\partial U} u, \tag{33}$$
and if $u(x_0) = \max_{\bar{U}} u$ for some $x_0 \in U$, then $u \equiv u(x_0)$;
- u superharmonic, then
$$\inf_{\bar{U}} u \geqslant \inf_{\partial U} u, \tag{34}$$
and if $u(x_0) = \inf_{\bar{U}} u$ for some $x_0 \in U$, then $u \equiv u(x_0)$.

This can be applied to obtain various estimates for Laplace's and Poisson's equations. For example, we can prove the estimate

$$\sup_{B_{1/2}} |D u| \leqslant C \sup_{\partial B_1} |u| \tag{35}$$

for harmonic function u.

To see this, we take a "cut-off" function $\eta \in C_0^1(B_1)$ such that $\eta \equiv 1$ in $B_{1/2}$. Then we compute

$$\begin{aligned} \triangle\left(\eta^2 |D u|^2\right) &= 2\,\eta \triangle \eta\, |D u|^2 + 2\,|D\eta|^2 |D u|^2 + 8\,\eta\,(D\eta) \cdot D^2 u \cdot D u + 2\,\eta^2 \left|D^2 u\right|^2 \\ &\geqslant 2\,\eta \triangle \eta\, |D u|^2 + 2\,|D\eta|^2 |D u|^2 - \left[8\,|D\eta|^2 |D u|^2 + 2\,\eta^2 \left|D^2 u\right|^2\right] + 2\,\eta^2 \left|D^2 u\right|^2 \\ &= \left(2\,\eta \triangle \eta - 6\,|D\eta|^2\right) |D u|^2 \\ &\geqslant -C\,|D u|^2. \end{aligned} \tag{36}$$

Next we notice that, if $\triangle u = 0$, then

$$\triangle \left(u^2 \right) = 2 \left| D u \right|^2. \tag{37}$$

As a consequence, we have

$$\triangle \left(\eta^2 \left| D u \right|^2 + \alpha\, u^2 \right) \geqslant 0 \tag{38}$$

for some constant α.

Thus $\eta^2 \left| D u \right|^2 + \alpha\, u^2$ is subharmonic, and we have

$$\max_{B_{1/2}} \left| D u \right|^2 \leqslant \max_{\overline{B_1}} \eta^2 \left| D u \right|^2 + \alpha\, u^2 \leqslant \max_{\partial B_1} \eta^2 \left| D u \right|^2 + \alpha\, u^2 = \alpha \left(\max_{\partial B_1} u \right)^2. \tag{39}$$

Remark 24. Note that the above argument does not involve mean value formula. Thus this method is more robust than estimating through mean value formula.