TRANSPORT EQUATIONS

We will consider the transport equation

$$\begin{cases} u_t + \boldsymbol{b} \cdot Du = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{1}$$

When $\boldsymbol{b} = \boldsymbol{b}(x, t)$, $f = f(x, t)$ the system is linear; When $\boldsymbol{b} = \boldsymbol{b}(x, t, u)$ and/or $f = f(x, t, u)$ the system is quasi-linear. Of course $\boldsymbol{b}$, f may depend on derivatives of u. We will not discuss those cases in this lecture.

1. Simplest case.

We start with the simplest case: $\boldsymbol{b}$ is a constant, $f = 0$.

The idea is to find the formula of solutions by transforming the equation into an ODE. To do this, we consider the equation along a curve

$$x = x(s), \quad t = t(s). \tag{2}$$

Along this curve, we can define a function of one variable s:

$$\tilde{u}(s) := u(x(s), t(s)). \tag{3}$$

Differentiating, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} \tilde{u}(s) = \dot{t}(s)\, u_t + \dot{x}(s) \cdot Du. \tag{4}$$

This would help us in transforming the PDE into ODE if

$$\dot{x}(s) = \boldsymbol{b}\, \dot{t}(s). \tag{5}$$

It is clear that we have the freedom here to choose $\dot{t}(s)$. We choose $\dot{t}(s) = 1$. Thus we have

$$t(s) = t_0 + s, \qquad x(s) = x_0 + \boldsymbol{b}\, s. \tag{6}$$

The equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}s} \tilde{u}(s) = 0. \tag{7}$$

As a consequence, we have

$$u(x_0, t_0) = \tilde{u}(0) = \tilde{u}(-t_0) = u(x_0 - \boldsymbol{b}\, t_0, 0) = g(x_0 - \boldsymbol{b}\, t_0). \tag{8}$$

Since the choice of (x_0, t_0) is arbitrary, we see that the solution formula is

$$u(x, t) = g(x - t\, \boldsymbol{b}). \tag{9}$$

Remark 1. The classical "algorithm" of solving 1st order PDEs starts with the "chain"

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x_1}{b_1} = \cdots = \frac{\mathrm{d}x_n}{b_n}. \tag{10}$$

This gives n first-integrals

$$x_1 - b_1\, t, \ldots, x_n - b_n\, t. \tag{11}$$

As a consequence, the general solution of

$$u_t + \boldsymbol{b} \cdot Du = 0 \tag{12}$$

is

$$u = F(x_1 - b_1\, t, \ldots, x_n - b_n\, t). \tag{13}$$

Comparing with the initial value we see that $F = g$ and

$$u(x, t) = g(x - t\, \boldsymbol{b}). \tag{14}$$

Remark 2. The "transport" effect is clear by considering the case where the initial value g is a "bump" supported in a ball in $\mathbb{R}^n$. Then the equation "moves" this ball with speed $\boldsymbol{b}$.

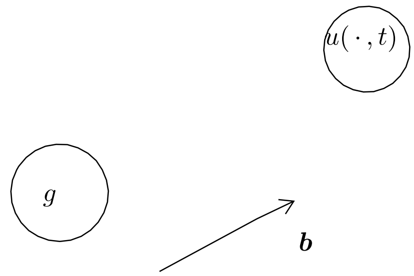

Figure 1. The $n=2$ case.

2. $\boldsymbol{f \neq 0}$.

We consider the problem

$$u_t + \boldsymbol{b} \cdot Du = f(x,t) \quad \text{in } \mathbb{R}^n \times (0,\infty); \qquad u = g \quad \text{on } \mathbb{R}^n \times \{0\}. \tag{15}$$

Let $x = x_0 + s\,\boldsymbol{b}, t = t_0 + s$, and

$$\tilde{u}(s) := u(x_0 + s\,\boldsymbol{b}, t_0 + s), \tag{16}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\tilde{u} = f(x_0 + s\,\boldsymbol{b}, t_0 + s). \tag{17}$$

Setting $s = 0$ and $-t_0$ respectively, we have

$$\begin{aligned} u(x_0, t_0) &= \tilde{u}(0) \\ &= \tilde{u}(-t_0) + \int_{-t_0}^{0} \frac{\mathrm{d}}{\mathrm{d}s}\tilde{u}(s)\,\mathrm{d}s \\ &= g(x_0 - t_0\,\boldsymbol{b}) + \int_{-t_0}^{0} f(x_0 + \boldsymbol{s}\,b, t_0 + s)\,\mathrm{d}s \\ &= g(x_0 - t_0\,\boldsymbol{b}) + \int_0^{t_0} f(x_0 + (s - t_0)\,\boldsymbol{b}, s)\,\mathrm{d}s. \end{aligned} \tag{18}$$

Realizing the arbitrariness of (x_0, t_0), we see that the formula for the solution is

$$u(x,t) = g(x - t\,\boldsymbol{b}) + \int_0^t f(x + (s-t)\,\boldsymbol{b}, s)\,\mathrm{d}s. \tag{19}$$

Remark 3. If we set

$$u_1(x,t) := g(x - t\,\boldsymbol{b}), \qquad u_2(x,t) := \int_0^t f(x + (s-t)\,\boldsymbol{b}, s)\,\mathrm{d}s, \tag{20}$$

then u_1 solves

$$u_t + \boldsymbol{b} \cdot Du = 0, \quad u = g \tag{21}$$

while u_2 solves

$$u_t + \boldsymbol{b} \cdot Du = f, \quad u = 0. \tag{22}$$

This is an illustration of the superposition principle for linear PDEs.

Remark 4. We check u_2 more carefully. We have

$$u_2(x,t) = \int_0^t f(x + (s-t)\,\boldsymbol{b}, s)\,\mathrm{d}s = \int_0^t f(x - (t-s)\,\boldsymbol{b}, s)\,\mathrm{d}s. \tag{23}$$

We realize that $f(x - (t-s)\,\boldsymbol{b}, s)$ is the value of the solution of

$$w_t + \boldsymbol{b} \cdot Dw = 0 \quad \text{in } \mathbb{R}^n \times (s,\infty); \qquad w(x,s) = f(x,s) \tag{24}$$

at time t. We denote such w by $w(x, t; s)$ to emphasize its dependence on s. This is an illustration of the so-called Duhamel's principle, which says once we have a formula of the initial value problem of a homogeneous evolutions equation, we can simply integrate in time to obtain the solution for the non-homogeneous equation.

For example, the solution to the non-homogeneous heat equation

$$u_t - \triangle u = f, \qquad u\,|_{t=0} = 0 \tag{25}$$

can be written as

$$u(x, t) = \int_0^t w(x, t; s)\,\mathrm{d}s \tag{26}$$

where $w(x, t; s)$ solves

$$w_t - \triangle_x w = 0, \qquad w(x, s; s) = f(x, s). \tag{27}$$

3. Interlude: Well-posedness.

We pause to consider the well-posedness of the equation

$$u_t + \boldsymbol{b} \cdot Du = f, \qquad u\,|_{t=0} = g \tag{28}$$

where $\boldsymbol{b}$ is a constant.

- Existence: The formula

$$u(x, t) = g(x - t\,\boldsymbol{b}) + \int_0^t f(x + (s - t)\,\boldsymbol{b}, s)\,\mathrm{d}s \tag{29}$$

  gives a C^1 function as long as, for example, g, f are both C^1. It is easy to verify that this function indeed satisfies the equation and the initial value. Thus existence is guaranteed.

- Uniqueness: It suffices to show that

$$u_t + \boldsymbol{b} \cdot Du = 0, \qquad u|_{t=0} = 0 \tag{30}$$

  has only 0 solution. To see this, we define $\tilde{u}$ as above. Then it is clear that $\frac{\mathrm{d}}{\mathrm{d}s}\tilde{u} = 0$. Since $\tilde{u}(-t_0) = 0$, we conclude that $\tilde{u} \equiv 0$. Thus shows uniqueness.

- Continuous dependence on data: Consider

$$u(x, t) = g(x - t\,\boldsymbol{b}) + \int_0^t f(x + (s - t)\,\boldsymbol{b}, s)\,\mathrm{d}s \tag{31}$$

  and

$$\hat{u}(x, t) = \hat{g}(x - t\,\boldsymbol{b}) + \int_0^t \hat{f}\left(x + (s - t)\,\hat{\boldsymbol{b}}, s\right)\mathrm{d}s. \tag{32}$$

  Taking the difference we have

$$u - \hat{u} = g - \hat{g} + \int_0^t f(x + (s - t)\,\boldsymbol{b}, s) - \hat{f}\left(x + (s - t)\,\hat{\boldsymbol{b}}, s\right)\mathrm{d}s. \tag{33}$$

  It is easy to see that as long as $f(x, t)$ is Lipschitz in the x variable, then $u - \hat{u}$ is bounded by $C\left[|g - \hat{g}| + \left|f - \hat{f}\right| + \left|\boldsymbol{b} - \hat{\boldsymbol{b}}\right|\right]$ which leads to continuous dependence.

4. Non-constant $\boldsymbol{b}$.

Consider

$$u_t + \boldsymbol{b}(x, t) \cdot Du = 0, \qquad u\,|_{t=0} = g. \tag{34}$$

Set $x = x(s), t = t(s)$, we see that the equation can be simplified if

$$\dot{x}(s) = b(x(s), t(s))\,\dot{t}(s). \tag{35}$$

This gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\dot{x}(s)}{\dot{t}(s)} = b(x, t). \tag{36}$$

Such a curve $x = x(t)$ is called characteristic. Along such a curve (same as saying setting $\tilde{u}(s) = u(x(t), t)$), the equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{u} = 0 \tag{37}$$

which gives

$$u(x(t), t) = \tilde{u}(x(0), 0) = g(x(0)). \tag{38}$$

Or, written another way,

$$u(x, t) = g(x_0(x, t)) \tag{39}$$

where $x_0(x, t)$ is such that the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = \boldsymbol{b}(X, t), \qquad X(0) = x_0(x, t) \tag{40}$$

satisfies

$$X(t) = x. \tag{41}$$

Remark 5. The formula shows that the equation still "transports", but with deformation.

Remark 6. Using Duhamel's principle, we can write down that formula for the non-homogeneous problem

$$u_t + \boldsymbol{b}(x, t) \cdot Du = f(x, t), \qquad u|_{t=0} = g(x) \tag{42}$$

as

$$u(x, t) = g(x_0(x, t)) + \int_0^t f(x_s(x, t), s)\, \mathrm{d}s \tag{43}$$

where $x_s(x, t)$ is such that if

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = \boldsymbol{b}(X, t), \qquad X(s) = x_s(x, t) \tag{44}$$

then

$$X(t) = x. \tag{45}$$

Remark 7. One popular way to show uniqueness is the "energy method", which proceeds as follows. We multiply the equation

$$u_t + \boldsymbol{b} \cdot Du = 0 \tag{46}$$

by u and then integrate over $\mathbb{R}^n$. We obtain

$$\int u\, u_t + \int (\boldsymbol{b} \cdot Du)\, u = 0 \tag{47}$$

Now some integration by parts gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int u^2\right] = -\int (\nabla \cdot \boldsymbol{b})\,\frac{u^2}{2} \leqslant \sup_{x,t} |\nabla \cdot \boldsymbol{b}| \left[\frac{1}{2}\int u^2\right]. \tag{48}$$

Now if $\frac{1}{2}\int u^2 = 0$ at $t = 0$, necessarily it has to be 0 at later times.

The energy method is easy to use and almost universally applicable, with one catch: $\int u^2$ must be finite. Thus this method cannot be used to show uniqueness when $u(x, 0)$ does not decay to 0 as $x \to \pm\infty$.

5. Well-posedness for the case $\boldsymbol{b} \neq$ constant.

- Existence: Guaranteed by ODE theory, as long as $\boldsymbol{b}$ is Lipschitz in x.
- Uniqueness: Guaranteed by ODE theory, as long as $\boldsymbol{b}$ is Lipschitz in x.
- Continuous dependence: If $u, \tilde{u}$ and solutions to the equation with data $\boldsymbol{b}$, g and $\tilde{\boldsymbol{b}}$, $\tilde{g}$ respectively, taking the difference $e := u - \tilde{u}$ we reach

$$e_t + \boldsymbol{b} \cdot De = \left(\tilde{\boldsymbol{b}} - \boldsymbol{b}\right) \cdot D\tilde{u}, \qquad e|_{t=0} = g - \tilde{g} \tag{49}$$

which gives

$$e(x,t) = (g - \tilde{g})(x_0(x,t)) + \int_0^t \left(\tilde{\boldsymbol{b}} - \boldsymbol{b}\right) \cdot D\tilde{u}(x_s(x,t), s)\, \mathrm{d}s. \tag{50}$$

From this we see that

1. Non-homogeneous equations need to be understood when studying homogeneous equations;
2. Estimates of $D\,\tilde{u}$ is needed. In other words, to study well-posedness, we need to study regularity.

- Regularity: From the formula it is clear that, if g, f are smooth, then u is as regular as $x_0(x,\,t)$. Standard ODE theory tells us that the regularity of x_0 is the same as that of $\boldsymbol{b}$.

6. An example of quasi-linear case.

In the quasi-linear case, regularity of data no longer leads to regularity of the solution. For example, consider the 1D Burgers equation

$$u_t + u\,u_x = 0, \qquad u\,|_{t=0} = g. \tag{51}$$

The method of characteristics leads to

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = u(x(t),t), \qquad x\,|_{t=0} = x_0, \qquad u(x(t),t) = g(x_0). \tag{52}$$

Since $u(x(t),t)$ is a constant, the characteristics $x(t)$ are straight lines. As a consequence, two characteristics with different slope may intersect. However since the slope is simply $1/u$, "different slope" implies the values of u along the two characteristics are different. Thus the solution cannot be defined at the intersection. In other words, the solution cease to be a function in finite time, no matter how smooth g is.