MATH 527 A1 HOMEWORK 6 (DUE DEC. 2 IN CLASS)

Exercise 1. (10 pts) (6.6.1)

Let

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} + cu \tag{1}$$

Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\ ,\ ]$ satisfies the hypothesis of the Lax − Milgram theorem, provided

$$c(x) \geqslant -\mu \qquad (x \in U). \tag{2}$$

Exercise 2. (10 pts) (6.6.5)

Let u be a smooth solution of $Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} = 0$ in U. Set $v := |Du|^2 + \lambda u^2$. Show that

$$Lv \leqslant 0 \quad \text{in } U, \text{ if } \lambda \text{ is large enough.} \tag{3}$$

Deduce

$$\|Du\|_{L^\infty(U)} \leqslant C\left(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)}\right). \tag{4}$$

Exercise 3. (10 pts) (6.6.9) (Courant minimax principle). Let $L = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j}$, where $\left(\left(a^{ij}\right)\right)$ is symmetric. Assume the operator L, with zero boundary conditions, has eigenvalues $0 < \lambda_1 < \lambda_2 \leqslant \cdots$. Show

$$\lambda_k = \max_{S \in \Sigma_{k-1}} \min_{u \in S^\perp, \|u\|_{L^2} = 1} B[u, u] \qquad (k = 1, 2, \ldots). \tag{5}$$

Here Σ_{k-1} denotes the collection of $(k-1)$-dimensional subspaces of $H_0^1(U)$.