MATH 527 A1 HOMEWORK 5 (DUE NOV. 18 IN CLASS)

Exercise 1. (10 pts) (5.10.6) Prove directly that if $u \in W^{1,p}(0,1)$ for some $1 < p < \infty$, then $|u(x) - u(y)| \leqslant |x - y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p \, \mathrm{d}t \right)^{1/p}$ for a.e. $x, y \in [0,1]$.

Exercise 2. (5 pts) (5.10.7) Denote by U the open square $\{x \in \mathbb{R}^2 \mid |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1 & x_1 > 0, |x_2| < x_1 \\ 1 + x_1 & x_1 < 0, |x_2| < -x_1 \\ 1 - x_2 & x_2 > 0, |x_1| < x_2 \\ 1 + x_2 & x_2 < 0, |x_1| < -x_2 \end{cases}. \tag{1}$$

For which $1 \leqslant p \leqslant \infty$ does u belong to $W^{1,p}(U)$?

Exercise 3. (10 pts) (5.10.8) Integrate by parts to prove the interpolation inequality

$$\int_U |Du|^2 \, \mathrm{d}x \leqslant C \left(\int_U u^2 \, \mathrm{d}x \right)^{1/2} \left(\int_U |D^2 u|^2 \, \mathrm{d}x \right)^{1/2} \tag{2}$$

for all $u \in C_c^\infty(U)$. Assume ∂U is smooth, and prove this inequality if $u \in H^2(U) \cap H_0^1(U)$. (Hint: Take $\{v_k\} \subset C_c^\infty$ converging to u in $H_0^1(U)$, and $\{w_k\} \subset C^\infty(\bar{U})$ converging to u in $H^2(U)$.)

Exercise 4. (5 pts) (5.10.13) Verify that if $n > 1$, the unbounded function $u = \log\log\left(1 + \frac{1}{|x|}\right)$ belongs to $W^{1,n}(U)$, for $U = B^0(0,1)$.

Exercise 5. (Optional) (5.10.10) Suppose U is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0 \qquad a.e. \text{ in } U. \tag{3}$$

Prove u is constant a.e. in U.