Math 527 A1 Homework 4 (Due Nov. 4 in Class)

Exercise 1. (5 pts) Let u be a weak solution of the scalar conservation law. Show that if $u \in C^1(\Omega)$ for some domain Ω, then it is a classical solution in Ω, that is

$$u_t + f(u)_x = 0 \text{ for } (x,t) \in \Omega, \qquad u(x,0) = u_0 \text{ when } (x,0) \in \Omega. \tag{1}$$

Exercise 2. (5 pts) Consider the Burgers equation with initial data $u_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$. Prove using definition that $u(x,t) = \begin{cases} 1 & x < t/2 \\ 0 & x > t/2 \end{cases}$ is a weak solution. Show that it is furthermore an entropy solution.

Exercise 3. (10 pts) (Evans 3.5.14)

Compute explicitly the unique entropy solution of

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \quad \text{in } \mathbb{R} \times (0,\infty); \qquad u = g \quad \text{on } \mathbb{R} \times \{t = 0\}. \tag{2}$$

for

$$g(x) = \begin{cases} 1 & x < -1 \\ 0 & -1 < x < 0 \\ 2 & 0 < x < 1 \\ 0 & x > 0 \end{cases}. \tag{3}$$

Exercise 4. (10 pts) (Evans 4.7.3) Consider the viscous conservation law

$$u_t + F(u)_x - a\, u_{xx} = 0 \qquad \text{in } \mathbb{R} \times (0,\infty) \tag{4}$$

where $a > 0$ and F is uniformly convex.

i. **(3 pts)** Show u solves (4) if $u(x,t) = v(x - \sigma t)$ and v is defined implicitly by the formula

$$s = \int_c^{v(s)} \frac{a}{F(z) - \sigma z + b}\, \mathrm{d}z \qquad (s \in \mathbb{R}), \tag{5}$$

where b and c are constants.

ii. **(3 pts)** Demonstrate that we can find a traveling wave satisfying

$$\lim_{s \to -\infty} v(s) = u_l, \qquad \lim_{s \to \infty} v(s) = u_r \tag{6}$$

for $u_l > u_r$, if and only if

$$\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}. \tag{7}$$

iii. **(4 pts)** Let u^ε denote the above traveling wave solution of (4) for $a = \varepsilon$, with $u^\varepsilon(0,\, 0) = \frac{u_l + u_r}{2}$. Compute $\lim_{\varepsilon \to 0} u^\varepsilon$ and explain your answer.