# MATH 527 A1 HOMEWORK 3 (DUE OCT. 21 IN CLASS)

**Exercise 1. (8 pts) (Evans 2.5.17)** Let $u \in C^2(\mathbb{R} \times [0, \infty))$ solve the initial-value problem for the wave equation in one dimension:

$$u_{tt} - u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, \infty); \qquad u = g,\ u_t = h \quad \text{on } \mathbb{R} \times \{t = 0\}. \tag{1}$$

Suppose $g, h$ have compact support. the *kinetic energy* is $k(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t)\, \mathrm{d}x$ and the *potential energy* is $p(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t)\, \mathrm{d}x$. Prove

i. **(4 pts)** $k(t) + p(t)$ is constant in $t$.

ii. **(4 pts)** $k(t) = p(t)$ for all large enough times $t$.

**Exercise 2. (2 pts) (Evans 3.5.3 b)** Solve using characteristics:

$$u\, u_{x_1} + u_{x_2} = 1, \qquad u(x_1, x_1) = \frac{1}{2} x_1. \tag{2}$$

**Exercise 3. (10 pts) (Evans 3.5.5)** Write $L = H^*$, if $H: \mathbb{R}^n \mapsto \mathbb{R}$ is convex.

a) **(5 pts)** Let $H(p) = \frac{1}{r} |p|^r$, for $1 < r < \infty$. Show

$$L(q) = \frac{1}{s} |q|^s, \qquad \text{where } \frac{1}{r} + \frac{1}{s} = 1. \tag{3}$$

b) **(5 pts)** Let $H(p) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}\, p_i\, p_j + \sum_{i=1}^{n} b_i\, p_i$, where $A = ((a_{ij}))$ is a symmetric, positive definite matrix, $b \in \mathbb{R}^n$. Compute $L(q)$.

**Exercise 4. (Optional) (Evans 3.5.6)** Let $H: \mathbb{R}^n \mapsto \mathbb{R}$ be convex. We say $q$ belongs to the *subdifferential* of $H$ at $p$, written

$$q \in \partial H(p) \tag{4}$$

if

$$H(r) \geqslant H(p) + q \cdot (r - p) \qquad \text{for all } r \in \mathbb{R}^n. \tag{5}$$

Prove $q \in \partial H(p)$ if and only if $p \in \partial L(q)$ if and only if $p \cdot q = H(p) + L(q)$, where $L = H^*$.

**Exercise 5. (10 pts) (Evans 3.5.8)** Let $E$ be a closed subset of $\mathbb{R}^n$. Show that if the Hopf-Lax formula could be applied to the initial-value problem

$$u_t + |Du|^2 = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty); \qquad u = \begin{cases} 0 & x \in E \\ +\infty & x \notin E \end{cases} \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \tag{6}$$

it would give the solution

$$u(x, t) = \frac{1}{4t} \operatorname{dist}(x, E)^2. \tag{7}$$