Math 527 A1 Homework 2 (Due Sep. 30 in Class)

Exercise 1. (2 pts) (Evans 2.5.2) Prove that Laplace's equation $\triangle u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) := u(O\,x) \qquad (x \in \mathbb{R}^n), \tag{1}$$

then $\triangle v = 0$.

Exercise 2. (6 pts) Prove the mean value formula for harmonic functions using Poisson's formula for the ball (see Evans 2.2.4c for the formula).

Exercise 3. (6 pts)

a) **(Evans 2.5.3)** Modify the proof of the mean value formulas to show for $n \geqslant 3$ that

$$u(0) = \frac{1}{|\partial B_r|} \int_{\partial B_r} g\,\mathrm{d}S + \frac{1}{n\,(n-2)\,\alpha(n)} \int_{B_r} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f\,\mathrm{d}x. \tag{2}$$

b) **(Optional)** Prove the above using Green's function for the ball B_r.

Exercise 4. (6 pts) (Evans 2.5.5) Prove that there exists a constant C, depending only on n, such that

$$\max_{B_1} |u| \leqslant C \left(\max_{\partial B_1} |g| + \max_{B_1} |f| \right) \tag{3}$$

whenever u is a smooth solution of

$$-\triangle u = f \text{ in } B_1; \qquad u = g \text{ on } \partial B_1. \tag{4}$$

Exercise 5. (4 pts) (Evans 2.5.10) Suppose u is smooth and solves $u_t - \triangle u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

i. **(1 pt)** Show $u_\lambda(x,t) := u(\lambda\,x, \lambda^2\,t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

ii. **(3 pts)** Use (i) to show $v(x,t) := x \cdot Du(x,t) + 2\,t\,u_t(x,t)$ solves the heat equation as well.

Exercise 6. (6 pts) (Evans 2.5.13) Given g: $[0,\infty) \mapsto \mathbb{R}$, with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\,\pi}} \int_0^t \frac{1}{(t-s)^{3/2}}\, e^{\frac{-x^2}{4(t-s)}}\, g(s)\,\mathrm{d}s \tag{5}$$

for a solution of the initial/boundary-value problem

$$u_t - u_{x\,x} = 0 \text{ in } \mathbb{R}_+ \times (0,\infty); \qquad u = 0 \text{ on } \mathbb{R}_+ \times \{t = 0\}; \qquad u = g \text{ on } \{x = 0\} \times [0, \infty). \tag{6}$$

(Hint: Let $v(x,t) := u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflections.)

Exercise 7. (Optional) (Evans 2.5.8) Let u be the solution of

$$\triangle u = 0 \text{ in } \mathbb{R}^n_+; \qquad u = g \text{ on } \partial \mathbb{R}^n_+ \tag{7}$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial \mathbb{R}^n_+$, $|x| \leqslant 1$. Show Du is not bounded near $x = 0$. (Hint: Estimate $\frac{u(\lambda\,e_n) - u(0)}{\lambda}$).

(For those who know the following stuff: This is the unboundedness of Riesz operators on L^∞ in disguise.)

Exercise 8. (Optional) Prove that the Dirac delta function $\delta(x)$ in $\mathbb{R}^n$ becomes

$$\frac{1}{n\,\alpha(n)} \frac{\delta(r)}{r^{n-1}} \tag{8}$$

in polar coordinates.