(Nov. 3, 3pm – 4:30pm, CAB457)

Math 527 (2008) Midterm

Problem 1. For any bounded, continuous function $g:\mathbb{R}\mapsto\mathbb{R}$ we define a function $f:\mathbb{R}^2\mapsto\mathbb{R}$ by

$$f(x_1,x_2)=\begin{cases}1 & x_2>g(x_1)\\ 0 & x_2<g(x_1)\end{cases}. \tag{1}$$

Let $T_f\in\mathcal{D}'(\mathbb{R}^2)$ be the distribution corresponding to f.

a) **(10 pts)** Compute the distributional derivative $\partial_2 T_f$ (∂_2 means $\frac{\partial}{\partial x_2}$).

b) **(10 pts)** Now let $g_n(x)=n^{-\alpha}\sin(n\,x)$ with $\alpha>0$, and define $f_n(x_1,x_2)$ in the above manner. Let T_n be the distribution corresponding to f_n. Find out the limit of the distributions $\partial_2 T_n$ as $n\nearrow\infty$ and prove convergence.

c) **(Extra 10 pts)** Does the limiting behavior remain the same when $\alpha=0$? If your answer is yes, prove your claim; If your answer is no, construct a counter-example.

Solution.

a) Let $\phi(x_1,x_2)\in C_0^\infty(\mathbb{R}^2)$ be a test function. Then by definition of distributional derivatives we have

$$\begin{aligned}(\partial_2 T_f)(\phi) &= -T_f(\partial_2\phi)\\ &= -\int_{\mathbb{R}^2} f(x_1,x_2)\,(\partial_2\phi(x_1,x_2))\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &= -\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x_1,x_2)\,(\partial_2\phi(x_1,x_2))\,\mathrm{d}x_2\right]\mathrm{d}x_1\\ &= -\int_{-\infty}^{\infty}\left[\int_{g(x_1)}^{\infty}\partial_2\phi(x_1,x_2)\,\mathrm{d}x_2\right]\mathrm{d}x_1\\ &= -\int_{-\infty}^{\infty}-\phi(x_1,g(x_1))\,\mathrm{d}x_1\\ &= \int_{\mathbb{R}}\phi(x_1,g(x_1))\,\mathrm{d}x_1.\end{aligned} \tag{2}$$

b) First for the particular g_n's ,we have

$$(\partial_2 T_n)(\phi)=\int_{\mathbb{R}}\phi(x,g_n(x))\,\mathrm{d}x. \tag{3}$$

Now note that $g_n(x)\to 0$ uniformly on $\mathbb{R}$, which means $\phi(x,\,g_n(x))\to\phi(x,\,0)$ uniformly too. Thus

$$(\partial_2 T_n)(\phi)=\int_{\mathbb{R}}\phi(x,g_n(x))\,\mathrm{d}x\to\int_{\mathbb{R}}\phi(x,0)\,\mathrm{d}x. \tag{4}$$

c) The answer is no. For a counter-example, recall that we have $\sin^2 n\,x\to\frac{1}{2}\neq 0$ in $\mathcal{D}'(\mathbb{R})$. Thus we construct a function $\phi(x_1,\,x_2)\in C_0^\infty$ such that it is bounded $|\phi|\leqslant 1$, and equals x_2^2 in the "box" $[-R,R]\times[-1,1]$ and vanishes outside $[-(R+1),(R+1)]\times[-2,2]$, and finally $\phi(x_1,0)=0$ for all x_1. We see that

$$(\partial_2 T_n)(\phi)=\int_{-R}^{R}\sin^2 n\,x\,\mathrm{d}x+O(1)\not\to 0=\int_{\mathbb{R}}\phi(x,0)\,\mathrm{d}x \tag{5}$$

when we take R large enough.

Remark 1. This is an special case of the theory of Young measures which deals with the behavior of nonlinear functionals under weak convergence and has been successfully applied to calculus of variations, conservation laws, control theory, etc. For our T_n, in fact one can show that

$$(\partial_2 T_n)(\phi) \to \int_{\mathbb{R}} \left[\int_{-1}^{1} \phi(x,z)\,\mathrm{d}\nu(z) \right] \mathrm{d}x \tag{6}$$

where $\mathrm{d}\nu$ is the following measure on $[-1,1]$:

$$\mathrm{d}\nu(z) = \frac{\mathrm{d}z}{\pi\sqrt{1-z^2}}. \tag{7}$$

In other words, we have the following convergence in the space of measures

$$\delta_{x_2 = g(x_1)} \to \chi_{\mathbb{R}\times[-1,1]}\,\mathrm{d}\nu(x_2)\,\mathrm{d}x_1. \tag{8}$$

For completeness we sketch how to obtain $\mathrm{d}\nu(z)$ here. Write

$$\begin{aligned}(\partial_2 T_n)(\phi) &= \int_{\mathbb{R}} \phi(x, \sin n\,x)\,\mathrm{d}x \\ &= \sum_k \int_{\frac{2k\pi}{n}}^{\frac{2(k+1)\pi}{n}} \phi(x,\sin n\,x)\,\mathrm{d}x \\ &= \sum_k \int_{\frac{2k\pi}{n}}^{\frac{2(k+1)\pi}{n}} \phi\left(\frac{2\,k\,\pi}{n}, \sin n\,x\right)\mathrm{d}x + R_n \\ &= \sum_k \frac{2\,\pi}{n}\left[\int_{-1}^{1} \phi\left(\frac{2\,k\,\pi}{n}, z\right)\mathrm{d}\nu(z)\right] + R_n \\ &= \int_{\mathbb{R}}\int_{-1}^{1} \phi(x,z)\,\mathrm{d}\nu(z)\,\mathrm{d}x + R_n + R_n' \end{aligned} \tag{9}$$

Here the last equality is because the sum can be seen as a Riemann sum. It is easy to show that both R_n, R_n' tends to 0 as $n\nearrow\infty$.

Problem 2. Let Ω be a bounded, connected domain with smooth boundary. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic.

a) **(10 pts)** State and prove the mean value formula for u.

b) **(10 pts)** State and prove the strong maximum principle.

Proof. See lecture notes. □

Problem 3. Let Ω be a bounded domain with smooth boundary. $u \in W_0^{1,2}(\Omega)$ is said to be an eigenfunction of the operator $-\triangle$ with Dirichlet boundary condition if u is a weak solution of

$$-\triangle u = \lambda\,u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{10}$$

for some $\lambda \in \mathbb{R}$. This λ is called the corresponding eigenvalue.

a) **(10 pts)** Prove that for any $\Omega' \subset\subset \Omega$ and any $k \in \mathbb{N}$, $u \in W^{k,2}(\Omega')$.

b) **(Extra 5 pts)** Prove that $u \in C^\infty(\Omega)$.

c) **(10 pts)** Prove that if λ is an eigenvalue, then $\lambda > 0$.

Solution.

a) From the regularity theory of weak solutions for $-\triangle u = f$, we know

$$\|u\|_{W^{k,2}(\Omega')} \leqslant C\left[\|u\|_{L^2(\Omega)} + \|f\|_{W^{k-2,2}(\Omega)}\right] \tag{11}$$

where C depends on Ω', Ω and k. Now in our case $f = \lambda u \in W^{1,2}(\Omega)$. This means $u \in W^{3,2}(\Omega')$ for any $\Omega' \subset\subset \Omega$ (and therefore automatically in $W^{2,2}(\Omega')$ too). Now for any $\Omega' \subset\subset \Omega$, we can find Ω'' such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Then $f = \lambda u \in W^{3,2}(\Omega'') \Longrightarrow u \in W^{5,2}(\Omega')$. Iterate, we can cover all k.

One can also make this argument rigorous by using induction (similar to the exercise we have done).

b) For any Ω' and any k, we have $u \in W^{k,2}(\Omega')$. Embedding gives $u \in C^m(\Omega')$ for any $m < k - \frac{n}{2}$ where n is the dimension. The arbitrariness of k implies $u \in C^\infty(\Omega')$, then the arbitrariness of Ω' gives $u \in C^\infty(\Omega)$.

c) There are at least two ways of showing this.

- Method 1. Maximum principle.

  Since $u \in C^\infty$, it is a classical solution. We know that the weak maximum principle holds for

$$-\triangle u + c(x)\, u = 0 \tag{12}$$

  when $c(x) \geqslant 0$. Thus the weak maximum principle holds when $\lambda \leqslant 0$. As a consequence, the only solution to

$$-\triangle u = \lambda u, \qquad u|_{\partial\Omega} = 0 \tag{13}$$

  is 0 and therefore such λ is not an eigenvalue. (I forgot to put $u \neq 0$ in the problem. Points will be added back if this is the only reason why you weren't able to finish the proof.)

- Method 2. Variational formulation.

  Since $u \in W_0^{1,2}(\Omega)$, it can be used as a test function in the weak formulation, that is

$$\int |\nabla u|^2 = -\int (\triangle u)\, u = \int (\lambda u)\, u = \lambda \int u^2 \tag{14}$$

  From this it is clear that $\lambda > 0$ unless $\nabla u = 0$. But in that case u is a constant. The proof is ended by noticing that the only constant function in $W_0^{1,2}(\Omega)$ is 0.

Problem 4. (Extra 10 pts). Let $u \geqslant 0$ be harmonic in the whole space $\mathbb{R}^n$. Prove that u must be a constant.

Proof. Fix any $x \neq y$. We will show $u(x) = u(y)$. Take any $r > \frac{|x-y|}{2}$, we have by mean value formula

$$\begin{aligned}
|u(x) - u(y)| &= \frac{1}{|B_r|}\left| \int_{B_r(x)} u - \int_{B_r(y)} u \right| \\
&= \frac{1}{|B_r|} \int_{B_r(x) \triangle B_r(y)} u \quad \text{(Because } u \geqslant 0) \\
&\leqslant \frac{1}{|B_r|} \int_{B_{r+\frac{|x-y|}{2}}\left(\frac{x+y}{2}\right) \backslash B_{r-\frac{|x-y|}{2}}\left(\frac{x+y}{2}\right)} u \\
&= \frac{1}{|B_r|}\left[\int_{B_{r+\frac{|x-y|}{2}}\left(\frac{x+y}{2}\right)} u - \int_{B_{r-\frac{|x-y|}{2}}\left(\frac{x+y}{2}\right)} u \right] \\
&= \frac{1}{|B_r|}\left[u\left(\frac{x+y}{2}\right) \left|B_{r+\frac{|x-y|}{2}}\right| - u\left(\frac{x+y}{2}\right) \left|B_{r-\frac{|x-y|}{2}}\right| \right] \\
&= u\left(\frac{x+y}{2}\right) \frac{\left(r + \frac{|x-y|}{2}\right)^n - \left(r - \frac{|x-y|}{2}\right)^n}{r^n} \\
&\rightarrow 0 \qquad \text{as } r \nearrow \infty.
\end{aligned} \tag{15}$$

Note that we have used the mean value formula twice!

- Method 2. (Suggested by Niksirat). First using mean value formula (or maximum principle) we see that if $u(x)=0$ for some x, then $u\equiv 0$.

  Now take any two points x, y. We modify the proof of the Harnack inequality to show that $u(y)/u(x)=1$. Take $R>|x-y|$. We have

$$u(x)=\frac{1}{|B_R|}\int_{B_R(x)} u \geqslant \frac{1}{|B_R|}\int_{B_{R-|x-y|}(y)} u = \frac{|B_{R-|x-y|}|}{|B_R|} u(y). \tag{16}$$

  Taking $R\nearrow\infty$ we conclude $u(x)\geqslant u(y)$. Switching the role of x and y we have $u(y)\geqslant u(x)$. Thus ends the proof. $\square$