## SEMIGROUP METHOD

In this lecture we establish properties of the heat equation through the abstract theory of semigroups. Let $u(x,t)$ be a solution to the heat equation. Then we have

1. $u(x,t_1+t_2)$ is the same as $v(t_2)$ where $v$ solves the heat equation with initial value $u(x,t_1)$;
2. $\lim_{t\searrow t_0} u(x,t)=u(x,t_0)$.

If we define a family of operators $T_t$ by

$$T_t v(x)=u(x,t) \tag{1}$$

where $u(x,t)$ solves the heat equation with initial value $v(x)$, then one can check:

1. $T_0 v=v$ for any $v$;
2. $T_{t_1+t_2}v=T_{t_2}(T_{t_1}v)$ for all $t_1,t_2\geqslant 0$;
3. $\lim_{t\to t_0} T_t v=T_{t_0}v$ for all $v$.

This motivates the following definition.

**Definition 1. (Continuous semigroup of operators)** *Let $B$ be a Banach space, and for $t>0$, let $T_t$: $B\mapsto B$ be continuous linear operators with*

*i.* $T_0=\mathrm{Id}$*;*

*ii.* $T_{t_1+t_2}=T_{t_2}\circ T_{t_1}$ *for all* $t_1,t_2\geqslant 0$*;*

*iii.* $\lim_{t\to t_0} T_t v=T_{t_0}v$ *for all* $t_0\geqslant 0$ *and all* $v\in B$.[1]

*Then the family* $\{T_t\}_{t\geqslant 0}$ *is called a continuous*[2] *semigroup (of operators).*

**Example 2.** In the case of the heat equation, one can take the space $B$ to be the space of bounded uniformly continuous functions $C_b^0$, with the norm

$$\|u\|_{C_b^0}=\sup|u|. \tag{3}$$

Note that from the maximum principle, we have furthermore

$$\|T_t u\|_{C_b^0}\leqslant\|u\|_{C_b^0} \tag{4}$$

for all $t\geqslant 0$. A semigroup with this extra property is called "**contracting**".

### 1. Infinitesimal generators.

The whole theory of semigroups is modeled after the theory of linear constant-coefficient ODE systems:

$$\dot{u}-A\,u=0, \qquad u(0)=u_0. \tag{5}$$

One can show that a family of operators $\{T_t\}$ defined by $T_t u_0=u(t)$ is a continuous semigroup. Note that the property of $T_t$ is determined by the matrix $A$.

Now we try to recover $A$ from $T_t$. We write

$$A\,u_0=\dot{u}(0)=\lim_{t\searrow 0}\frac{1}{t}\,[u(t)-u(0)]=\lim_{t\searrow 0}\frac{1}{t}\,[T_t u_0-T_0 u_0]=\lim_{t\searrow 0}\frac{1}{t}\,[T_t-T_0]\,u_0. \tag{6}$$

This holds for any $u_0$, therefore

$$A=\lim_{t\searrow 0}\frac{1}{t}\,[T_t-T_0]. \tag{7}$$

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*1. One may be tempted to require*

$$\lim_{t\to t_0} T_t=T_{t_0}, \tag{2}$$

*but this requirement would be too strong severely restrict the application of the resulting theory.*

*2. The "continuous" here refers to the fact that $T_t$ is continuous with respect to the parameter $t$.*

We try to do the same thing for general continuous semigroups.

**Definition 3. (Infinitesimal generators)** *Let* $\{T_t\}_{t \geqslant 0}$ *be a continuous semigroup on a Banach space* $B$*. We put*

$$D(A) \equiv \left\{ v \in B \colon \lim_{t \searrow 0} \frac{1}{t} (T_t - \mathrm{Id}) v \text{ exists} \right\} \subset B \tag{8}$$

*and call the linear operator*

$$A \colon\ D(A) \mapsto B, \tag{9}$$

*defined as*

$$A v \equiv \lim_{t \searrow 0} \frac{1}{t} (T_t - \mathrm{Id}) v, \tag{10}$$

*the infinitesimal generator of the semigroup* $\{T_t\}$*.*

**Remark 4.** In general, $D(A)$ is not $B$.

Also note that the limit always exists for $v = 0$. Therefore $D(A)$ is never empty.

$D(A)$ has the following property.

**Lemma 5.** *For all* $v \in D(A)$*, and all* $t \geqslant 0$*, we have*

$$T_t A v = A T_t v. \tag{11}$$

*That is* $A$ *commutes with all* $T_t$*'s.*

**Proof.** For $v \in D(A)$, we have

$$T_t A v = T_t \lim_{s \searrow 0} \frac{1}{s} (T_s - \mathrm{Id}) v = \lim_{t \searrow 0} \frac{1}{s} [T_{t+s} - T_t] v = \lim_{s \searrow 0} \frac{1}{s} (T_s - \mathrm{Id})\, T_t v = A T_t v. \tag{12}$$

Here the second equality used the fact that $T_t$ is continuous and linear. □

**2. Contracting semigroups.**

Let $\{T_t\}$ be a contracting semigroup, that is $\{T_t\}$ is a continuous semigroup, and furthermore

$$\|T_t v\| \leqslant \|v\|. \tag{13}$$

Let $A$ be its infinitesimal generator. We want to show that $D(A)$ is dense in $B$. We do this through constructing for any $v \in B$ a sequence $\{J_\lambda v\} \subset D(A)$ such that $J_\lambda v \to v$ in $B$ as $\lambda \nearrow \infty$, where

$$J_\lambda v \equiv \int_0^\infty \lambda\, e^{-\lambda s}\, T_s\, v\, \mathrm{d}s \tag{14}$$

is well-defined for $\lambda > 0$.[3]

1. $\|J_\lambda v\| \leqslant \|v\|$.

   To see this, compute

$$\begin{aligned} \|J_\lambda v\| &= \left\| \int_0^\infty \lambda\, e^{-\lambda s} T_s\, v\, \mathrm{d}s \right\| \\ &\leqslant \int_0^\infty \lambda\, e^{-\lambda s}\, \|T_s v\|\, \mathrm{d}s \\ &\leqslant \int_0^\infty \lambda\, e^{-\lambda s}\, \|v\|\, \mathrm{d}s \quad \text{(Recall } T_s \text{ is contracting)} \\ &= \|v\| \int_0^\infty \lambda\, e^{-\lambda s}\, \mathrm{d}s \\ &= \|v\|. \end{aligned} \tag{15}$$

2. $J_\lambda v \to v$ in $B$. That is

$$\|J_\lambda v - v\| \to 0 \qquad \text{as } \lambda \nearrow \infty. \tag{16}$$

3. See J. Jost **Partial Differential Equations**, pp. 130–131 for details.

We compute

$$\begin{aligned}\|J_\lambda v - v\| &= \left\| \int_0^\infty \lambda\, e^{-\lambda s} T_s\, v\, \mathrm{d}s - v \right\| \\ &= \left\| \int_0^\infty \lambda\, e^{-\lambda s} (T_s\, v - v)\, \mathrm{d}s \right\|. \end{aligned} \tag{17}$$

For any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|T_s\, v - v\| < \frac{\varepsilon}{2}, \qquad \forall 0 \leqslant s \leqslant \delta. \tag{18}$$

Thus we write

$$\begin{aligned}\|J_\lambda v - v\| &= \left\| \int_0^\infty \lambda\, e^{-\lambda s} (T_s\, v - v)\, \mathrm{d}s \right\| \\ &\leqslant \left\| \int_0^\delta \lambda\, e^{-\lambda s} (T_s\, v - v)\, \mathrm{d}s \right\| + \left\| \int_\delta^\infty \lambda\, e^{-\lambda s} (T_s\, v - v)\, \mathrm{d}s \right\| \\ &\leqslant \int_0^\delta \lambda\, e^{-\lambda s} \|T_s\, v - v\|\, \mathrm{d}s + \int_\delta^\infty \lambda\, e^{-\lambda s} [\|T_s v\| + \|v\|]\, \mathrm{d}s \\ &\leqslant \frac{\varepsilon}{2} \int_0^\delta \lambda\, e^{-\lambda s}\, \mathrm{d}s + 2\, \|v\| \int_\delta^\infty \lambda\, e^{-\lambda s}\, \mathrm{d}s \\ &\leqslant \frac{\varepsilon}{2} + 2\, \|v\|\; e^{-\lambda \delta}. \end{aligned} \tag{19}$$

The RHS is less than $\varepsilon$ for all $\lambda > \lambda_0 \equiv \delta^{-1} \log(4\, \|v\| / \varepsilon)$.

3. For $v \in B$, $J_\lambda v \in D(A)$ for all $\lambda > 0$.

   We need to show the limit

$$\lim_{t \searrow 0} \frac{1}{t} (T_t - \mathrm{Id}) J_\lambda v \tag{20}$$

   exists.

   Compute

$$\begin{aligned}\frac{1}{t} (T_t - \mathrm{Id}) J_\lambda v &= \frac{1}{t} (T_t - \mathrm{Id}) \int_0^\infty \lambda\, e^{-\lambda s} T_s\, v\, \mathrm{d}s \\ &= \frac{1}{t} \int_0^\infty \lambda\, e^{-\lambda s}\, T_{t+s}\, v\, \mathrm{d}s - \frac{1}{t} \int_0^\infty \lambda\, e^{-\lambda s} T_s\, v\, \mathrm{d}s \\ &= \frac{1}{t} \int_t^\infty \lambda\, e^{\lambda t}\, e^{-\lambda s'}\, T_{s'} v\, \mathrm{d}s' - \frac{1}{t} \int_0^\infty \lambda\, e^{-\lambda s} T_s\, v\, \mathrm{d}s \\ &= \frac{e^{\lambda t} - 1}{t} \int_t^\infty \lambda\, e^{-\lambda s} T_s\, v\, \mathrm{d}s - \frac{1}{t} \int_0^t \lambda\, e^{-\lambda s} T_s\, v\, \mathrm{d}s. \end{aligned} \tag{21}$$

   It is clear that the limits exist for both terms. More specifically, as $t \searrow 0$, we have

$$\frac{e^{\lambda t} - 1}{t} \int_t^\infty \lambda\, e^{-\lambda s} T_s\, v\, \mathrm{d}s \longrightarrow \lambda\, J_\lambda v; \qquad \frac{1}{t} \int_0^t \lambda\, e^{-\lambda s} T_s\, v\, \mathrm{d}s \longrightarrow \lambda\, v. \tag{22}$$

   Therefore $J_\lambda v \in D(A)$, and furthermore

$$A\, J_\lambda\, v = \lambda\, (J_\lambda - \mathrm{Id})\, v. \tag{23}$$

4. Combining the above, we have the following theorem.

   **Theorem 6.** *Let $\{T_t\}_{t \geqslant 0}$ be a contracting semigroup with infinitesimal generator $A$. then $D(A)$ is dense in $B$.*

**Remark 7.** One can also define the (two-sided) derivative of $T_t$ at time $t > 0$: $D_t T_t$: $D(D_t T_t) \mapsto B$ by setting

$$(D_t T_t) v \equiv \lim_{h \to 0} \frac{1}{h} (T_{t+h}\, v - T_t\, v) \tag{24}$$

and $D(D_tT_t)$ is the subspace of $B$ in which the above limit exists.

It turns out that such a definition is not necessary, due to the following lemma.

**Lemma 8.** *$v \in D(A)$ implies $v \in D(D_tT_t)$, and furthermore*

$$D_tT_tv = AT_t\, v = T_t\, A v. \tag{25}$$

**Proof.** For any $v \in D(A)$, we have

$$\lim_{h \searrow 0} \frac{1}{h}\,(T_{t+h} - T_t)v = \lim_{h \searrow 0} \frac{1}{h}(T_h - \mathrm{Id})\, T_tv = AT_t\, v. \tag{26}$$

Thus the right derivative exists.

For the left derivative, we write $(h > 0)$

$$\frac{1}{-h}\,(T_{t-h} - T_t)\, v - T_t\, A v = T_{t-h}\left[\frac{1}{h}\,(T_h - I)\, v - A\, v\right] + (T_{t-h} - T_t)\, A\, v \to 0 \tag{27}$$

as $h \searrow 0$, since $T_{t-h}$ is uniformly bounded and $T_t\, A\, v$ is continuous with respect to $t$. □

## 3. The resolvent.

The resolvent is defined by

$$R(\lambda, A) \equiv (\lambda\, \mathrm{Id} - A)^{-1}. \tag{28}$$

We have the following results.

**Theorem 9.** *Let $A$ be the infinitesimal generator of a contracting semigroup. For $\lambda > 0$, the operator $(\lambda\, \mathrm{Id} - A)^{-1}$ is invertible, and we have*

$$(\lambda\, \mathrm{Id} - A)^{-1} = R(\lambda, A) = \frac{1}{\lambda}\, J_\lambda, \tag{29}$$

*that is,*

$$R(\lambda, A)\, v = \int_0^\infty e^{-\lambda s}\, T_s\, v\, \mathrm{d}s. \tag{30}$$

**Proof.**

- We show first that $\lambda\, \mathrm{Id} - A$ is injective (one-to-one), which implies that $(\lambda\, \mathrm{Id} - A)^{-1}$ is well-defined.

  Since $\lambda\, \mathrm{Id} - A$ is a linear operator, it suffices to show that there is no nonzero $v \in D(A)$ such that $(\lambda\, \mathrm{Id} - A)\, v = 0$, or equivalently $A\, v = \lambda\, v$, for $\lambda > 0$.

  For this particular $v$, we have

  $$D_tT_t\, v = T_t\, A v = \lambda\, (T_t\, v) \tag{31}$$

  which implies

  $$T_tv = e^{\lambda t}\, v \tag{32}$$

  This obviously contradicts the assumption that $\{T_t\}$ is contracting.

- We need to show

  $$(\lambda\, \mathrm{Id} - A)^{-1} v = \frac{1}{\lambda}\, J_\lambda\, v \tag{33}$$

  or equivalently

  $$(\lambda\, \mathrm{Id} - A)\, J_\lambda\, v = \lambda\, v \tag{34}$$

  for any $v \in B$. But this is just (23).

- Combining the above, we have show

  1. $(\lambda\, \mathrm{Id} - A)$ is one-to-one on $D(A)$;
  2. $(\lambda\, \mathrm{Id} - A)$ maps the image of $J_\lambda$ to the whole Banach space $B$.

This two facts force the image of $J_\lambda$ to be exactly $D(A)$ and consequently $(\lambda\,\mathrm{Id}-A)$ is bijective from $D(A)$ to $B$. $\square$

**Lemma 10.** **(Resolvent equation)** *Under the same assumption as the above theorem, we have for* $\lambda$*,* $\mu > 0$*,*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)\, R(\lambda, A)\, R(\mu, A). \tag{35}$$

**Proof.** Omitted. $\square$

## 4. Hille-Yosida Theorem.

**Theorem 11.** **(Hille-Yosida)** *Let* $A: D(A) \mapsto B$ *be a linear operator whose domain of definition* $D(A)$ *is dense in the Banach space* $B$*. Suppose that the resolvent* $R(n, A) = (n\,\mathrm{Id} - A)^{-1}$ *exists for all* $n \in \mathbb{N}$*, and that*

$$\left\| \left( \mathrm{Id} - \frac{1}{n} A \right)^{-1} \right\| \leqslant 1 \qquad \text{for all } n \in \mathbb{N}, \tag{36}$$

*Then* $A$ *generates a unique contracting semigroup.*

**Proof.** We just sketch the proof. For details see J. Jost **Partial Differential Equations**, pp. 139–142.

Let

$$J_n \equiv \left( \mathrm{Id} - \frac{1}{n} A \right)^{-1}. \tag{37}$$

1. We first show that

$$\lim_{n \nearrow \infty} J_n v = v \qquad \forall v \in B. \tag{38}$$

   Recall that $D(A)$ is dense in $B$, therefore since $\|J_n\| \leqslant 1$ uniformly it suffices to show this for all $v \in D(A)$.

   For such $v$ we compute

$$J_n\, A v = J_n\, (A - n\,\mathrm{Id}) v + n\, J_n v = n\, (J_n - \mathrm{Id})\, v. \tag{39}$$

   Thus

$$J_n\, v - v = \frac{J_n\, A\, v}{n} \to 0 \tag{40}$$

   again due to the uniform bound on $J_n$.

2. Since $\|J_n\| \leqslant 1$, we define

$$T_t^{(n)} \equiv \exp(-t\, n) \exp(t\, n\, J_n) = \exp(t\, A\, J_n) \tag{41}$$

   which is a contracting semigroup. The plan now is to show that $T_t^{(n)}$ converges to the desired semigroup (Note that $A\, J_n \to A$ strongly so this plan makes sense).

3. For any $u \in D(A)$,

$$\begin{aligned} \left\| T_t^{(n)} v - T_t^{(m)} v \right\| &= \left\| \int_0^t D_t\!\left( T_{t-s}^{(m)}\, T_s^{(n)}\, v \right) \mathrm{d}s \right\| \\ &= \left\| \int_0^t T_{t-s}^{(m)}\, T_s^{(n)}\, [A\, J_n - J_n\, A] v\, \mathrm{d}s \right\| \\ &\leqslant t\, \|(A\, J_n - J_m\, A) v\| \\ &= t\, \|(J_n - J_m)(A v)\| \end{aligned} \tag{42}$$

   Thus for each $v \in D(A)$, $\left\{ T_t^{(n)} v \right\}$ is a Cauchy sequence. One can further conclude that this also holds for all $v \in B$.

4. Now define $T_t$ to be the limit. We claim that it is a continuous contracting semigroup.

$$\begin{aligned}\|T_{t+s}v - T_t T_s v\| &\leqslant \left\|T_{t+s}v - T_{t+s}^{(n)}v\right\| + \left\|T_t^{(n)}T_s^{(n)}v - T_t^{(n)}T_s v\right\| \\ &\quad + \left\|T_t^{(n)}T_s v - T_t T_s v\right\| \longrightarrow 0.\end{aligned} \tag{43}$$

5. Next we show that the infinitesimal generator is $A$. Let $v \in D(A)$, we have

$$\begin{aligned}\lim_{t\searrow 0}\frac{1}{t}(T_t v - v) &= \lim_{t\searrow 0}\frac{1}{t}\lim_{n\nearrow\infty}\left(T_t^{(n)}v - v\right) \\ &= \lim_{t\searrow 0}\frac{1}{t}\lim_{n\nearrow\infty}\int_0^t T_s^{(n)} A\, J_n\, v\,\mathrm{d}s \\ &= \lim_{t\searrow 0}\frac{1}{t}\int_0^t T_s\, A v\,\mathrm{d}s \\ &= Av.\end{aligned} \tag{44}$$

Thus if $\bar{A}$ is the infinitesimal generator of $T_t$, we have $D(A) \subset D(\bar{A})$ and $\bar{A} = A$ in $D(A)$.

We now show $D(\bar{A}) = D(A)$. Take any $n > 0$. By assumption we know $n\,\mathrm{Id} - A$ is a bijection from $D(A)$ to $B$. On the other hand, since $\tilde{A}$ is the generator of a contracting semigroup, $n\,\mathrm{Id} - \tilde{A}$ is a bijection from $D\left(\tilde{A}\right)$ to $B$[4]. Therefore $D(A) = D\left(\tilde{A}\right)$.

6. Finally we show that such $T_t$ is unique.

Assume the contrary, that is there is $\tilde{T}_t$ with the same generator $A$. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t} T_s \tilde{T}_{t-s}\, v = A\, T_s \tilde{T}_{t-s}\, v - T_s\, A\, \tilde{T}_{t-s}\, v = 0. \tag{45}$$

Setting $s = 0$ and $t$ we obtain $T_t\, v = \tilde{T}_t\, v$. □

## 5. Application to the heat equation.

We would like to show that $A = \triangle$ satisfies the conditions in the Hille-Yosida theorem and thus there is a unique solution to the heat equation. We set $B$ to be the space of bounded, uniformly continuous functions.

All we need to show is that, if

$$\left(\mathrm{Id} - \frac{1}{n}\triangle\right)^{-1} f = g \Longleftrightarrow g - \frac{1}{n}\triangle g = f, \tag{46}$$

then

$$\sup|g| \leqslant \sup|f|, \tag{47}$$

Note that this is equivalent to

$$\sup g \leqslant \sup|f|, \qquad \sup(-g) \leqslant \sup|f| \tag{48}$$

Since $(-g) - \frac{1}{n}\triangle(-g) = (-f)$ and $|f| = |-f|$, it suffices to show

$$\sup g \leqslant \sup f. \tag{49}$$

There are two cases.

1. If $g$ attains its maximum at some point $x_0$, then $\triangle g(x_0) \leqslant 0$, which implies

$$\sup f \geqslant f(x_0) = g(x_0) - \frac{1}{n}\triangle g(x_0) \geqslant g(x_0) = \sup g. \tag{50}$$

2. If $g$ does not attain its maximum, we consider auxiliary functions

$$g_\varepsilon(x) = g(x) - \varepsilon\,|x|^2. \tag{51}$$

Since $g$ is bounded, $g_\varepsilon$ attains its maximum at some $x_\varepsilon$, where we have

$$\triangle g_\varepsilon(x_\varepsilon) \leqslant 0 \Longrightarrow \triangle g(x_\varepsilon) \leqslant 2\,d\,\varepsilon \tag{52}$$

4. See references for the proof of this.

Now for any $y$, by the choice of $x_\varepsilon$ we have

$$\begin{aligned} g(y) &\leqslant g_\varepsilon(y) + \varepsilon |y|^2 \\ &\leqslant g_\varepsilon(x_\varepsilon) + \varepsilon\ |y|^2 \\ &= g(x_\varepsilon) - \varepsilon\, |x_\varepsilon|^2 + \varepsilon\, |y|^2 \\ &\leqslant g(x_\varepsilon) + \varepsilon\, |y|^2 \\ &\leqslant g(x_\varepsilon) - \frac{1}{n} \triangle g(x_\varepsilon) + \varepsilon \left( \frac{2\,d}{m} + |y|^2 \right) \\ &= f(x_\varepsilon) + \varepsilon \left( \frac{2\,d}{n} + |y|^2 \right) \\ &\leqslant \sup f + \varepsilon \left( \frac{2\,d}{n} + |y|^2 \right). \end{aligned} \tag{53}$$

Taking $\varepsilon \searrow 0$ we obtain

$$g(y) \leqslant \sup f \text{ for all } y \implies \sup g \leqslant \sup f \tag{54}$$

and finish the proof.

**Further reading.**

- K.-J. Engel, R. Nagel, **One-Parameter Semigroups for Linear Evolution Equations**, Chapter 1 and the first 3 sections of Chapter 2.
- J. Jost, **Partial Differential Equations**, Chapter 6.

**Exercises.**

**Exercise 1.** Let $B$ be the space $\mathbb{R}^n$. Define a family of operators $\{T_t\}$ by $T_t u_0 = u(t)$ where $u$ solves

$$\dot{u} - A\,u = 0, \qquad u(0) = u_0. \tag{55}$$

Show that $\{T_t\}$ is a continuous semigroup. When is it contracting? For those $A$ generating a contracting semigroup, show directly that $\lambda\,\mathrm{Id} - A$ is invertible for all $\lambda > 0$.

**Exercise 2.** Let $B$ be $C_b^0(\mathbb{R})$. Define a family of operators $\{T_t\}$ by

$$(T_t\, f)(x) = f(x + t). \tag{56}$$

Show that $\{T_t\}$ is a continuous semigroup. Find its infinitesimal generator $A$ (meaning: both the formula for $A$ and the domain $D(A)$).