Viscosity Solutions

In this lecture we take a glimpse of the viscosity solution theory for linear and nonlinear PDEs. From our experience we know that even for linear equations, the existence of solutions is not easy to establish, where the best strategy is to first show the existence of solutions in a weaker sense and then establish regularity. However for a general nonlinear PDE, the usual strategy of multiplying by test functions does not work, and unless the equation is an Euler-Lagrange equation of some functional (that is, the equation is the necessary condition for u to be a minimizer of some functional), it is also not clear how variational formulation/direct methods can help.

It turns out that one can define a new type of weak solutions by "testing" the solution in a whole new sense. Such solutions are called "viscosity solutions", and a quite complete regularity theory which parallels those we have seen has been established in the past 25 years.[1]

1. Examples of fully nonlinear elliptic PDEs.

In this section we list a few example of the equations that can be dealt with using the idea of viscosity solutions.

Example 1. Hamilton-Jacobi equation.

$$H(x, u, Du) = 0 \tag{1}$$

In particular, the eikonal equation

$$|Du| = n(x) \tag{2}$$

for some $n(x) > 0$ in $\bar{\Omega}$. Here

$$|Du| = \left(\sum_i (\partial_i u)^2 \right)^{1/2}. \tag{3}$$

Example 2. Hamilton-Jacobi-Bellman equation.

Let

$$L^\alpha u \equiv \sum_{i,j=1}^N a_{ij}^\alpha(x)\, \partial_{ij} u + \sum_{i=1}^N b_i^\alpha(x)\, \partial_i u + c^\alpha(x)\, u(x) - f^\alpha(x) \tag{4}$$

where $A^\alpha \equiv (a_{ij}^\alpha)$ are positive semi-definite with a_{ij}^α only L^∞.

The Hamilton-Jacobi-Bellman equation reads

$$\sup_\alpha \{L^\alpha u\} = 0. \tag{5}$$

In particular, when α takes only one value, Lu is a second order elliptic operator which is not in divergence form (thus the Dirichlet principle cannot help since a_{ij}^α are not differentiable).

Example 3. Obstacle problem.

$$\max \left\{ F\big(x, u, Du, D^2 u\big), u - f(x) \right\} = 0, \tag{6}$$

or

$$\min \left\{ F\big(x, u, Du, D^2 u\big), u - f(x) \right\} = 0, \tag{7}$$

or

$$\max \left\{ \min \left\{ F\big(x, u, Du, D^2 u\big), u - f(x) \right\}, u - g(x) \right\} = 0. \tag{8}$$

Example 4. Monge-Ampère equation.

$$u \text{ is convex}, \quad \det\big(D^2 u\big) = f(x, u, Du). \tag{9}$$

1. Among the main contributors are L. Caffarelli, M. G. Crandall, L. C. Evans, H. Ishii, and P. L. Lions. In particular, the 1983 paper "Viscosity solutions of Hamilton-Jacobi equations" by Crandall and Lions is one of the contributions that helps the latter to win the Fields Medal in 1994.

2. Viscosity solutions.

Consider a function $F: \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n)^2 \mapsto \mathbb{R}$, written as $F(x, r, p, X)$. We call F

- degenerate elliptic if it is nonincreasing in its matrix argument:

$$F(x, r, p, X) \leqslant F(x, r, p, Y) \qquad \text{for} \qquad Y \leqslant X^3 \tag{10}$$

  and furthermore

- proper if it is also nondecreasing in r:

$$F(x, s, p, X) \leqslant F(x, r, p, Y) \qquad \text{for} \qquad s \leqslant r,\ Y \leqslant X. \tag{11}$$

Example 5. The (minus) Laplacian $-\triangle$ corresponds to $F(x, r, p, X) = -\operatorname{tr}(X)$. We can check that it is proper.

Example 6. The Hamilton-Jacobi equation corresponds to

$$F(x, r, p, X) = H(x, r, p). \tag{12}$$

It is proper as long as H is nondecreasing in r.

Definition 7. *Let F be proper, Ω be open and $u: \Omega \mapsto \mathbb{R}$. Consider the fully nonlinear equation*

$$F\left(x, u, Du, D^2 u\right) = 0. \tag{13}$$

Then u is a

- *viscosity subsolution if it is upper-semicontinuous*[4] *and for every $\varphi \in C^2(\Omega)$ and $\hat{x} \in \Omega$ which is a local maximum of $u - \varphi$, we have*

$$F\left(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})\right) \leqslant 0; \tag{18}$$

- *viscosity supersolution if it is lower-semicontinuous and for every $\varphi \in C^2(\Omega)$ and $\hat{x} \in \Omega$ which is a local minimum of $u - \varphi$, we have*

$$F\left(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})\right) \geqslant 0; \tag{19}$$

- *viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.*

Remark 8. By definition any viscosity solution is continuous.

Remark 9. To visualize the situation, notice that only $D\varphi$ and $D^2\varphi$ are involved, therefore one can freely replace φ by $\varphi + c$. Thus "$\hat{x}$ is a local maximum of $u - \varphi$" can be visualized as (the graph of) φ "touches" (the graph of) u from above and "$\hat{x}$ is a local minimum of $u - \varphi$" can be visualized as φ "touches" u from below.

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2. The space of $n \times n$ symmetric matrices.

3. Meaning $X - Y$ is positive semi-definite.

4. *A function u is upper semicontinuous if*

$$u(x) = u^*(x) \equiv \limsup_{r \searrow 0} \{u(y):\ y \in \Omega,\ |y - x| \leqslant r\}, \tag{14}$$

or equivalently

$$u(x) \geqslant \limsup_{k \nearrow \infty} u(x_k) \tag{15}$$

whenever $x_k \to x$; u is lower semicontinuous if

$$u(x) = u_*(x) \equiv \liminf_{r \searrow 0} \{u(y):\ y \in \Omega,\ |y - x| \leqslant r\}, \tag{16}$$

or equivalently

$$u(x) \leqslant \liminf_{k \nearrow \infty} u(x_k) \tag{17}$$

whenever $x_k \to x$. Note that if u is both lower semicontinuous and upper semicontinuous, then u is continuous.

Remark 10. The above definition generalizes the classical ones. For example, if $u \in C^2$ satisfies $F\big(x, u, Du, D^2 u\big) \leqslant 0$, then for any $\varphi \in C^2$, and $\hat{x} \in \Omega$ where $u - \varphi$ reaches maximum, then we have $D\varphi(\hat{x}) = Du(\hat{x})$ and $D^2\varphi(\hat{x}) \geqslant D^2 u(\hat{x})$ which implies

$$F\big(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})\big) \leqslant F\big(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2 u(\hat{x})\big) \leqslant 0 \tag{20}$$

since F is proper.

Example 11. Consider the equation

$$|u'|^2 = 1 \text{ in } (-1, 1), \qquad u(-1) = u(1) = 0. \tag{21}$$

Note that ambiguity appears when we try to formulate it into the form in the definition of viscosity solutions. We can use

$$F(x, r, p, X) = |p|^2 - 1. \tag{22}$$

or

$$G(x, r, p, X) = 1 - |p|^2. \tag{23}$$

We will explore this equation now. Along the way we will see that these two formulations are not equivalent in the viscosity sense.

- First note that our equation cannot have a classical solution in C^1. To see this, notice that since u' is continuous and $\int_{-1}^{1} u' = 0$, there must be a point $\xi \in (-1, 1)$ such that $u'(\xi) = 0$. Thus by continuity $|u'|^2 = 1$ cannot hold in some small neighborhood of ξ.
- Next we see that if we relax the requirement to $u \in \text{Lip}$ and satisfies $|u'|^2 = 1$ almost everywhere, then there are infinitely many solutions. These solutions are between $1 - |x|$ and $|x| - 1$.
- Then we can check that $1 - |x|$ is a viscosity solution to $F = 0$ and $|x| - 1$ is a solution to $G = 0$.
- Finally we show that $1 - |x|$ is the only viscosity solution to $F = 0$. In other words, we regain the lost uniqueness by consider viscosity solutions.

  To see this, let u be a viscosity solution of $F = 0$. That is, for any $\varphi \in C^2$ and any $\hat{x}$ maximizing (minimizing) $u - \varphi$, we have $|\varphi'(\hat{x})| \leqslant 1$ ($\geqslant 1$). We want to show that $u = 1 - |x|$.

  - Taking $\varphi = c\,(x - 1)$ with $c < -1$. We see that we must have $u \leqslant \varphi$. Taking $c \nearrow -1$ we see $u \leqslant 1 - x$.
  - Similarly we can show $u \leqslant x + 1$. Thus we have $u \leqslant 1 - |x|$.
  - On the other hand, consider the test function φ obtained by "smoothing out" the tip of $c\,(1 - |x|)$ for $0 < c < 1$. For such φ we have $u(\pm 1) = \varphi(\pm 1)$. Now if there is $\xi \in (-1, 1)$ such that $u(\xi) < \varphi(\xi)$, we know there must be a minimizer $\hat{x} \in (-1, 1)$ of $u - \varphi$. Contradiction.
  - Combining the above, we see that $u \leqslant 1 - |x|$ but $u \geqslant \varphi$ for φ arbitrarily close to $1 - |x|$. Thus $u = 1 - |x|$.

3. Maximum principles (comparison principles).

Let Ω be a bounded open set in $\mathbb{R}^n$. Consider the Dirichlet problem

$$H(x, u, Du) = 0 \text{ in } \Omega; \qquad u = g \text{ on } \partial\Omega. \tag{24}$$

Here H is continuous and proper on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ and $g \in C(\partial\Omega)$. We say that $u: \bar{\Omega} \mapsto \mathbb{R}$ is a subsolution if u is upper-semicontinuous and $H(\hat{x}, u(\hat{x}), Du(\hat{x})) \leqslant 0$ for any $\hat{x}$ maximizing $u - \varphi$, and $u \leqslant g$ on the boundary. We define supersolutions similarly.

We would like to show the comparison principle: When u is a subsolution and v is a supersolution, then $u \leqslant v$.

Remark 12. Note that uniqueness immediately follows from the comparison principle.

Remark 13. Note that uniqueness does not hold for all H. For example, let $w \in C^1(\bar{\Omega})$ be such that $w = 0$ on $\partial\Omega$. Then obviously $\pm w$ are both classical (and thus viscosity) solutions of

$$H(x, u, Du) \equiv |Du|^2 - |Dw|^2 = 0. \tag{25}$$

As a consequence, the comparison principle cannot hold for such H.

Now we discuss the idea of the proof. Pretending that u and v are smooth enough, we can use v as a test function for the subsolution u, and conclude: For any $\hat{x}$ such that $u - v$ reaches local maximum, $H(\hat{x}, u(\hat{x}), Dv(\hat{x})) \leqslant 0$. Using u as a test function for the supersolution v, noticing that the same $\hat{x}$ minimizes $v - u$, we obtain $H(\hat{x}, v(\hat{x}), Dv(\hat{x})) = H(\hat{x}, v(\hat{x}), Du(\hat{x})) \geqslant 0$. In particular, we have

$$H(\hat{x}, u(\hat{x}), Dv(\hat{x})) \leqslant H(\hat{x}, v(\hat{x}), Du(\hat{x})) \quad \Longrightarrow \quad u(\hat{x}) \leqslant v(\hat{x}) \tag{26}$$

as long as $H(x, r, p)$ is strictly increasing in r. Recalling that $\hat{x}$ is any local maximizer, and $u \leqslant v$ on the boundary, we conclude that $u \leqslant v$ everywhere.

The above argument is obviously flawed, as u and v may not be differentiable. Furthermore, recalling out example of $H = |p|^2 - 1$, it is not possible to remedy this by showing somehow the solutions are C^2.

The method to overcome this difficulty is the following trick of doubling the variables. Instead of using $v(x)$ as a test function, we use $v(y) + \frac{1}{2\varepsilon}|x - y|^2$, note that this function is smooth with respect to x. Let $(\hat{x}_\varepsilon, \hat{y}_\varepsilon)$ be the maximizer of

$$\Phi(x, y) \equiv u(x) - \left[v(y) + \frac{1}{2\varepsilon}|x - y|^2\right]. \tag{27}$$

Note that

$$u(x) - v(x) = \Phi(x, x) \leqslant \Phi(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \leqslant u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon) \tag{28}$$

for any $x \in \Omega$. Therefore all we need to do is to show that $\liminf_{\varepsilon \searrow 0} [u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon)] \leqslant 0$.

To show this, we use the fact that $\hat{x}_\varepsilon$ maximizes $u(x) - \left[v(\hat{y}_\varepsilon) + \frac{1}{2\varepsilon}|x - \hat{y}_\varepsilon|^2\right]$ and the fact that u is a subsolution to obtain

$$H\left(\hat{x}_\varepsilon, u(\hat{x}_\varepsilon), D_x\left[v(\hat{y}_\varepsilon) + \frac{1}{2\varepsilon}|x - \hat{y}_\varepsilon|^2\right]\right) \leqslant 0 \tag{29}$$

which simplifies to

$$H\left(\hat{x}_\varepsilon, u(\hat{x}_\varepsilon), \frac{x - \hat{y}_\varepsilon}{\varepsilon}\right) \leqslant 0. \tag{30}$$

On the other hand, $\hat{y}_\varepsilon$ minimizes $v(y) - \left[u(\hat{x}_\varepsilon) - \frac{1}{2\varepsilon}|\hat{x}_\varepsilon - y|^2\right]$ which leads to

$$H\left(\hat{y}_\varepsilon, v(\hat{y}_\varepsilon), \frac{x - \hat{y}_\varepsilon}{\varepsilon}\right) \geqslant 0. \tag{31}$$

Summarizing, we have in particular

$$H\left(\hat{x}_\varepsilon, u(\hat{x}_\varepsilon), \frac{x - \hat{y}_\varepsilon}{\varepsilon}\right) - H\left(\hat{y}_\varepsilon, v(\hat{y}_\varepsilon), \frac{x - \hat{y}_\varepsilon}{\varepsilon}\right) \leqslant 0. \tag{32}$$

At this stage one needs to restrict oneself to special cases. We will discuss the simplest one, when

$$H(x, r, p) = r + G(p) - f(x) \tag{33}$$

for $f \in C(\bar{\Omega})$ (and therefore uniformly continuous).

For such H, we obtain

$$u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon) \leqslant f(\hat{x}_\varepsilon) - f(\hat{y}_\varepsilon). \tag{34}$$

Recalling

$$u(x) - v(x) = \Phi(x, x) \leqslant \Phi(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \leqslant u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon), \tag{35}$$

we see that all we need to do is to show that $\hat{x}_\varepsilon - \hat{y}_\varepsilon \longrightarrow 0$ as $\varepsilon \searrow 0$.

This can be shown easily as follows. Recall that $(\hat{x}_\varepsilon, \hat{y}_\varepsilon)$ is the maximizer of $\Phi(x, y)$, we have in particular $\Phi(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \geqslant \Phi(x, x)$ which leads to

$$\frac{1}{2\,\varepsilon}\,|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2 \leqslant u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon) - u(x) + v(x) \leqslant \text{Constant}^5. \tag{36}$$

Remark 14. In Section 4 of M. G. Crandall, *Viscosity solutions: a primer*, a stronger result

$$\frac{1}{\varepsilon}\,|\hat{x}_\varepsilon - \hat{y}_\varepsilon|^2 \longrightarrow 0 \tag{37}$$

is proved. But as we have seen here, this stronger estimate is not necessary in our simple case.

Remark 15. The above result does not include the eikonal equation $|Du|^2 - f(x) = 0$, which we have indeed shown that cannot enjoy the comparison principle. But for the case $f > 0$ on $\bar{\Omega}$, one can set $v = -e^{-u}$ to obtain an equation for v which enjoys it. For a direct proof for the eikonal equation, see H. Ishii *A simple, direct proof of uniqueness for solutions of the Hamilton-Jacobi equations of eikonal type*, Proceedings of the AMS, 100(2), June 1987, 247 – 251.

4. Existence and Perron's method.

With the help of the comparison principle, one can show existence via Perron's method.

Theorem 16. (Ishii) *Let the comparison principle holds for the Dirichlet problem. Further assume that there is a subsolution $\underline{u}$ and a supersolution $\bar{u}$ which satisfy the boundary condition: $\underline{u} = \bar{u} = g$ on $\partial\Omega$. Then*

$$W(x) \equiv \sup\{w(x)\colon\ \underline{u} \leqslant w \leqslant \bar{u} \text{ and } w \text{ is a subsolution}\} \tag{38}$$

is a viscosity solution of the problem.

The basic idea is to show that if a subsolution w is not a solution, then one can modify it to obtain another subsolution $\tilde{w}$ such that $\tilde{w} > w$ in a small neighborhood. The details can be found in Section 9 of M. G. Crandall, *Viscosity solutions: a primer*.

5. Idea of regularity: Tangent paraboloids and second order differentiability.

I haven't been able to find a proof of the regularity for the Hamilton-Jacobi equation. Here we will just mention how it is possible to obtain regularity for viscosity solutions through properties of the test function φ. The material comes from §1.2 of the book L. A. Caffarelli, X. Cabré, **Fully Nonlinear Elliptic Equations**.

We consider test functions of the form

$$\varphi_\pm(x) = a + b \cdot x \pm \frac{M}{2}\,|x|^2 \tag{39}$$

where M is a positive constant.

Let $u \in C(\bar{\Omega})$. We try to "test" u from above using φ_+ with positive M and from below using φ_- with negative M. We say φ "touches" u from above(below) at x_0 in a subset A of Ω if

$$u(x) \leqslant \varphi(x)(\geqslant \varphi(x)) \qquad \forall x \in A; \qquad u(x_0) = \varphi(x_0). \tag{40}$$

In other words, x_0 is a local maximizer(minimizer) of $u - \varphi$.

Now set

$$\Theta(u, A)(x_0) \equiv \max\{\text{arginf}_M \varphi_+, \text{arginf}_M \varphi_-\}. \tag{41}$$

If it is not possible to "test" u by $\varphi_+(\varphi_-)$ from above(below) for any M, we say $\Theta(u, A)(x_0) = \infty$.

The crucial observation is the following: Let e be any unit direction. We define the second finite difference

$$\triangle_{h,e}^2 u(x_0) \equiv \frac{u(x_0 + h\,e) + u(x_0 - h\,e) - 2\,u(x_0)}{h^2}. \tag{42}$$

5. As x is an arbitrary but fixed point, $u(x) - v(x)$ is a constant. Now since u is upper-seimcontinuous (definition of subsolutions), $\max u$ is attained; Similarly $\min v$ is attained. Therefore $u - v$ attains a finite maximum.

Then it is easy to see that

$$\left|\triangle_{h,e}^2 u(x_0)\right| \leqslant \Theta\big(u, B_{|h|}(x_0)\big)(x_0). \tag{43}$$

From this one can obtain

Proposition 17. *Let $1 < p \leqslant \infty$ and u be a continuous function in Ω. Let ε be a positive constant and define*

$$\Theta(u,\varepsilon)(x) \equiv \Theta(u, \Omega \cap B_\varepsilon(x))(x), \qquad x \in \Omega. \tag{44}$$

Assume that $\Theta(u,\varepsilon) \in L^p$. Then $D^2 u \in L^p$ and

$$\left\|D^2 u\right\|_{L^p(\Omega)} \leqslant 2\, \|\Theta(u,\varepsilon)\|_{L^p(\Omega)}. \tag{45}$$

Furthermore, when Ω is convex and $\Theta(u,\varepsilon)$ is uniformly bounded, we can obtain $u \in C^{1,1}$ with

$$|Du(x) - Du(y)| \leqslant 2\, n\, \|\Theta(u,\varepsilon)\|_{L^\infty} |x-y|. \tag{46}$$

Further readings.

- M. G. Crandall, *Viscosity solutions: a primer*, Viscosity solutions and applications (Montecatini Terme, 1995), 1–43, Lecture Notes in Math., 1660, Springer, Berlin, 1997.
- M. G. Crandall, H. Ishii, P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bulletin of the AMS, 27(1), 1992, 1–67.
- L. A. Caffarelli, X. Cabré, **Fully Nonlinear Elliptic Equations**, AMS, 1995.

Exercises.

Exercise 1. (Regularity theory for the Laplace equation) Consider the Laplace equation $\triangle u = 0$. Show that if u is a viscosity solution, then u is harmonic (that is u is a classical solution).

Exercise 2. (Optional). Consider the equation

$$|Du| - 1 = 0, \qquad x \in \Omega \subset \mathbb{R}^n; \qquad u = 0 \quad x \in \partial\Omega. \tag{47}$$

Show that $u(x) \equiv \operatorname{dist}(x, \partial\Omega)$ solves this equation in the viscosity sense.