Theory of Linear Elliptic PDE

We will sketch, in this section, the theory of general linear elliptic PDEs:

$$L(u) \equiv \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial u(x)}{\partial x_i} + c(x)\, u(x) = f(x) \tag{1}$$

in some domain $\Omega \subset \mathbb{R}^n$. We make the following assumptions

a) Ellipticity: There is $\lambda > 0$ such that for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^{n} a^{ij}(x)\, \xi_i\, \xi_j \geqslant \lambda\, |\xi|^2. \tag{2}$$

We further assume $a^{ij} = a^{ji}$.

b) Boundedness: There exists $K < \infty$ such that

$$\left|a^{ij}(x)\right|, \left|b^i(x)\right|, |c(x)| \leqslant K \qquad \forall x \in \Omega. \tag{3}$$

c) (For Schauder estimates only) Hölder continuous coefficients: There exists $K < \infty$ such that

$$\left\|a^{ij}\right\|_{C^\alpha(\Omega)}, \left\|b^i\right\|_{C^\alpha(\Omega)}, \|c\|_{C^\alpha(\Omega)} \leqslant K \tag{4}$$

for all i, j.

1. Maximum principles.

We first note that, in the general case, the sign of $c(x)$ becomes important.

Example 1. Consider the 1D Dirichlet problem

$$u''(x) + u(x) = 0 \qquad \text{on } (0, \pi); \qquad u(0) = u(\pi) = 0, \tag{5}$$

which has $a \sin x$ as its solutions. Thus no maximum principle could possibly hold. Therefore we should not expect maximum principles when $c > 0$.

Theorem 2. *Assume $c(x) \equiv 0$, and let u satisfy in Ω*

$$L(u) \geqslant 0, \tag{6}$$

that is

$$\sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial u(x)}{\partial x_i} \geqslant 0, \tag{7}$$

then

$$\sup_{x \in \Omega} u(x) = \max_{x \in \partial\Omega} u(x). \tag{8}$$

In the case $L(u) \leqslant 0$, a corresponding result holds with $\sup / \max$ *replaced by* $\inf / \min$.

Proof. (Sketch).

1. Consider the case $L(u) > 0$. Let x_0 be an interior maximum. Then $\nabla u(x_0) = 0$ and $\nabla^2 u(x_0)$ negative semidefinite. Show that any symmetric matrix can be written as a sum of rank one matrices,[1] and obtain contradiction.

2. For the case $L(u) \geqslant 0$, consider the function $v_\varepsilon \equiv u + \varepsilon\, e^{\alpha x_1}$. And show that appropriate choices of α guarantees

$$L(v_\varepsilon) > 0 \tag{9}$$

1. A matrix $A = (a_{ij})$ is rank-one if there is a vector ξ such that $a_{ij} = \xi_i\, \xi_j$.

and then apply the first step. Finally take $\varepsilon \searrow 0$. □

Remark 3. A consequence is the uniqueness of solutions when $c(x) \equiv 0$.

Corollary 4. *Suppose $c(x) \leqslant 0$ in Ω. Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $L(u) \geqslant 0$ in Ω. Write $u^+(x) \equiv \max(u(x), 0)$, we then have*

$$\sup_{\Omega} u^+ \leqslant \max_{\partial\Omega} u^+. \tag{10}$$

Proof. Let $\Omega^+ = \{x \in \Omega \colon u(x) \geqslant 0\}$. Then apply the theorem. □

Now we turn to the strong maximum principle of E. Hopf.

Theorem 5. *Suppose $c(x) \equiv 0$, let u satisfy*

$$L(u) = 0 \qquad \text{in } \Omega. \tag{11}$$

If u attains its maximum in the interior of Ω, then it has to be constant.

If $c(x) \leqslant 0$, then u has to be a constant if it attains a nonnegative interior maximum.

Proof. (Sketch)

1. Assume by contradiction that u is not constant. Then

$$\Omega' \equiv \{x \in \Omega \colon u(x) < m \equiv \max u\} \neq \phi, \tag{12}$$

and

$$\partial\Omega' \cap \Omega \neq \phi. \tag{13}$$

2. Choose y such that there is r that $B_r(y) \subset \Omega'$ and $\partial B_r(y) \cap \partial\Omega' = \{x_0\} \subset \Omega$. Apply the following boundary point lemma of E. Hopf.

Lemma. *Suppose $c(x) \leqslant 0$ and*

$$L(u) \geqslant 0 \qquad \text{in } \Omega \subset \mathbb{R}^n, \tag{14}$$

and let $x_0 \in \partial\Omega$. Moreover, assume

i. u is continuous at x_0;

ii. $u(x_0) \geqslant 0$ if $c(x) \not\equiv 0$;

iii. $u(x_0) > u(x)$ for all $x \in \Omega$;

iv. there exists an open ball $B_r(y) \subset \Omega$ with $x_0 \in \partial B_r(y)$.

Then we have

$$\frac{\partial u}{\partial n}(x_0) > 0, \tag{15}$$

where n is the outer normal of the ball $B_r(y)$ at x_0, provided that this derivative exists.

3. Proof of the lemma **(sketch)**.

a. By taking a smaller ball, we can assume $B_r(y) \cap \partial\Omega = \{x_0\}$.

b. Consider $v(x) \equiv e^{-\gamma|x-y|^2} - e^{-\gamma r^2}$ on $B_r \backslash B_\rho$ for $0 < \rho < r$. Show $L(v) \geqslant 0$.

c. Find ε such that

$$w_\varepsilon(x) \equiv u(x) - u(x_0) + \varepsilon\, v(x) \leqslant 0, \qquad x \in \partial B_\rho. \tag{16}$$

d. Show that $L(w_\varepsilon) \geqslant 0$. And apply weak maximum principle. □

2. Schauder estimates.

Schauder estimates are generalizations of the $C^{2,\alpha}$ estimates of the Poisson equation $\triangle u = f$.

Theorem 6. *Let $f \in C^{\alpha}(\Omega)$, and suppose $u \in C^{2,\alpha}(\Omega)$ satisfies*

$$L u = f \tag{17}$$

in Ω with $0 < \alpha < 1$. Then for any $\Omega_0 \subset\subset \Omega$ we have

$$\|u\|_{C^{2,\alpha}(\Omega_0)} \leqslant C \left(\|f\|_{C^{\alpha}(\Omega)} + \|u\|_{L^2(\Omega)} \right). \tag{18}$$

where the constant depends on $\Omega, \Omega_0, \alpha, n, \lambda, K$.

Proof. (Sketch)

1. Note that when $b^i = 0$, $c = 0$, and a^{ij} are constants, one can obtain the estimate easily by doing a linear change-of-variables.
2. For $x_0 \in \overline{\Omega_0}$, one can write $L u = f$ in the following form:

$$\sum_{i,j} a^{ij}(x_0) \frac{\partial u(x)}{\partial x_i \partial x_j} = \varphi(x) \tag{19}$$

where

$$\varphi(x) = \sum_{i,j} \left(a^{ij}(x_0) - a^{ij}(x) \right) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \sum_i b^i(x) \frac{\partial u(x)}{\partial x_i} - c(x)\, u(x) + f(x). \tag{20}$$

3. Some computation yields

$$\|\varphi\|_{C^{\alpha}(B_R(x_0))} \leqslant \sup_{i,j,x \in B_R(x_0)} \left| a^{ij}(x_0) - a^{ij}(x) \right| \|u\|_{C^{2,\alpha}(B_R(x_0))} + C \|u\|_{C^2(B_R(x_0))} + \|f\|_{C^{\alpha}}. \tag{21}$$

4. The result of step 1 implies

$$\|u\|_{C^{2,\alpha}(B_r(x_0))} \leqslant C \left[\sup_{i,j,x \in B_R(x_0)} \left| a^{ij}(x_0) - a^{ij}(x) \right| \|u\|_{C^{2,\alpha}(B_R(x_0))} + C \|u\|_{C^2(B_R(x_0))} + \|f\|_{C^{\alpha}} \right]. \tag{22}$$

for some $r < R$.

5. Choose R small enough so that

$$\sup_{i,j,x \in B_R(x_0)} \left| a^{ij}(x_0) - a^{ij}(x) \right| \leqslant \frac{1}{2}. \tag{23}$$

6. Recall that for any $\varepsilon > 0$, there is $N(\varepsilon)$ such that

$$\|u\|_{C^2(B_R(x_0))} \leqslant \varepsilon \|u\|_{C^{2,\alpha}(B_R(x_0))} + N(\varepsilon) \|u\|_{L^2(B_R(x_0))}. \tag{24}$$

Finally note that only finitely many such balls are needed to cover $\overline{\Omega_0}$. □

Theorem 7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{2,\alpha}$. Let $f \in C^{\alpha}(\overline{\Omega})$ and $g \in C^{2,\alpha}(\overline{\Omega})$. Assume $u \in C^{2,\alpha}(\overline{\Omega})$ satisfy*

$$L u(x) = f(x) \qquad x \in \Omega; \qquad u(x) = g(x) \qquad x \in \partial\Omega. \tag{25}$$

Then

$$\|u\|_{C^{2,\alpha}(\Omega)} \leqslant C \left(\|f\|_{C^{\alpha}(\Omega)} + \|g\|_{C^{2,\alpha}(\Omega)} + \|u\|_{L^2(\Omega)} \right). \tag{26}$$

Here the constant depends on $\Omega, \alpha, d, \lambda$ and K.

Proof. (Sketch)

1. First let $u = u - g$ we make the boundary condition 0.

2. Locally we can stretch $\partial\Omega$ into a straight line using a $C^{2,\alpha}$ change of varaible.
3. Obtain the estimate for the problem

$$\triangle u = f \qquad \text{in } B_R^+, \qquad f \in C^\alpha\left(\overline{B_R^+}\right), \tag{27}$$

$$u = 0 \qquad \text{on } \partial^0 B_R^+. \tag{28}$$

for $0 < r < R$:

$$\|u\|_{C^{2,\alpha}\left(B_r^+\right)} \leqslant C\left(\|f\|_{C^\alpha\left(B_R^+\right)} + \|u\|_{L^2\left(B_R^+\right)}\right). \tag{29}$$

By considering $\varphi = \eta\, u$ for certain cut-off function η.
4. Finish the proof by a "frozen-coefficients" and "finite covering" argument. □

3. Weak solutions.

Weak solutions are easily defined for equations in the divergence form:

$$L(u) \equiv \sum_i \partial_i\left(\sum_j a^{ij}(x)\,\partial_j u(x) + b_i(x)\,u(x)\right) + c(x)\,u(x) = f(x), \qquad u = g \text{ on } \partial\Omega. \tag{30}$$

Definition 8. *$u \in W^{1,2}(\Omega)$ is a weak solution if $u - g \in W_0^{1,2}(\Omega)$ and for any $v \in W_0^{1,2}(\Omega)$,*

$$\int_\Omega \sum_{i,j} a^{ij}(x)\,\partial_j u(x)\,\partial_i v(x) + b_i(x)\,u(x)\,\partial_i v(x) + c(x)\,u(x)\,v(x)\,\mathrm{d}x + \int_\Omega f(x)\,v(x) = 0. \tag{31}$$

For such weak solutions, one can obtain a priori esitimates similar to that of the Poisson equation. On the other hand, the existence is more involved.