Regularity Theory for Weak Solutions

Recall that we have defined the weak solution u for the Poisson equation

$$\triangle u = f, \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega \tag{1}$$

by a function $u \in W^{1,2}$ such that $u - g \in W_0^{1,2}$ and

$$\int \nabla u \cdot \nabla v + \int f v = 0 \tag{2}$$

for all $v \in W_0^{1,2}(\Omega)$. We will discuss the regularity of u in this lecture. To do this, we need to survey the theory of spaces similar to $W^{1,2}$, that is, the Sobolev spaces.

1. Sobolev spaces.

Sobolev spaces are originally designed with the purpose of studying elliptic PDEs. In this section we only sketch its major properties. For details, see J. Jost **Partial Differential Equations**, §8.1. For more details, see L. Tartar *An Introduction to Sobolev spaces and interpolation spaces*, available online at http://www.math.cmu.edu/cna/publications.html, or R. A. Adams **Sobolev Spaces**, Academic Press.

Definition 1. *The Sobolev space $W^{k,p}(\Omega)$ contains all the distributions whose distributional derivatives are L^p integrable:*

$$D^\alpha u \in L^p(\Omega), \qquad \forall |\alpha| \leqslant k. \tag{3}$$

Remark 2. The space $L^p(\Omega)$ can be viewed from two perspects:

1. It contains distributions that are
   a) (for $p = 1$) integrable;
   b) (for $1 < p < \infty$) bounded in the following sense:
   $$T(\phi) \leqslant K \left(\int \phi^q \right)^{1/q} \tag{4}$$
   for all $\phi \in C_0^\infty(\Omega)$. Here $q = \frac{p}{p-1}$ (that is, $\frac{1}{p} + \frac{1}{q} = 1$);
   c) (for $p = \infty$) essentially bounded.
2. For $1 \leqslant p < \infty$, L^p is the completion of the space $C_0^\infty(\Omega)$ (or $C^\infty(\Omega)$) under the norm
   $$\|f\|_{L^p} \equiv \left(\int_\Omega f^p \right)^{1/p}. \tag{5}$$

The space $W^{k,p}(\Omega)$ has the following properties.

1. $W^{k,p}(\Omega)$ is complete with respect to the norm
   $$\|u\|_{W^{k,p}(\Omega)} \equiv \left(\sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha u|^p \right)^{1/p}. \tag{6}$$
   The Sobolev space $W_0^{k,p}(\Omega)$ is the completion of the set $C_0^\infty(\Omega)$ with respect to this norm.

   Note. For general Ω, $W_0^{k,p}(\Omega) = W^{k,p}(\Omega)$ only when $k = 0$. For other k's the two spaces are different.

2. $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$.
3. The most important properties of Sobolev spaces are the embedding theorems.

Theorem 3. (Sobolev embedding)

$$W_0^{1,p}(\Omega) \subset \left\{ \begin{array}{ll} L^{\frac{np}{n-p}} & p < n \\ C^0(\bar{\Omega}) & p > n \end{array} \right. . \tag{7}$$

Moreover we have

$$\|u\|_{L^{(np)/n-p}} \leqslant C \, \|\nabla u\|_{L^p}; \qquad \sup_{\Omega} |u| \leqslant C \, |\Omega|^{\frac{1}{n}-\frac{1}{p}} \, \|\nabla u\|_{L^p}. \tag{8}$$

Scaling. Instead of proving the theorem, we present a "back-of-envelope" way of remembering the formulas.

Denote by l the length scale and h the height scale. Then we have

$$\|u\|_{L^q} \sim (h^q \, l^n)^{1/q}; \qquad \|\nabla u\|_{L^p} \sim \left(\left(\frac{h}{l} \right)^p l^n \right)^{1/p}; \qquad \sup_{\Omega} |u| \sim h. \tag{9}$$

We see that when $q = \frac{n\,p}{n-p}$ (require $p < n$), the scaling of $\|u\|_{L^q}$ and $\|\nabla u\|_{L^p}$ are the same, therefore we expect an absolute constant (in particular, the constant C in

$$\|u\|_{L^{(np)/n-p}} \leqslant C \, \|\nabla u\|_{L^p} \tag{10}$$

is independent of Ω. It also works when $\Omega = \mathbb{R}^n$.[1]); When $p > n$, we have

$$\|\nabla u\|_{L^p} \sim h \, l^{(n-p)/p} \sim (\sup |u|) \, (l^n)^{1/p - 1/n}, \tag{11}$$

the relation

$$\sup_{\Omega} |u| \leqslant C \, |\Omega|^{\frac{1}{n}-\frac{1}{p}} \, \|\nabla u\|_{L^p} \tag{12}$$

is implied after realizing l^n is the scaling of the volumn $|\Omega|$.

Corollary. *Iterating, we have*

$$W_0^{k,p}(\Omega) \subset \left\{ \begin{array}{ll} L^{\frac{np}{n-kp}}(\Omega) & k\,p < n \\ C^m(\Omega) & 0 \leqslant m < k - \dfrac{n}{p} \end{array} \right. . \tag{13}$$

The same scaling argument can serve as intuition.

One can further refine the result and show that

$$H_0^{1,p}(\Omega) \subset C^{1-\frac{n}{p}}(\bar{\Omega}) \tag{14}$$

when $1 - \frac{d}{p}$ is not an integer.

2. L^2 regularity.

2.1. Interior regularity.

Our goal is the prove the following theorem, which justifies the intuition that u is twice more differentiable than f. By "interior regularity", we mean we do not deal with boundary data, and therefore the L^2-norm of u is necessary in the RHS.

Theorem 4. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\triangle u = f$ with $f \in L^2(\Omega)$. For any $\Omega' \subset\subset \Omega$, we have $u \in W^{2,2}(\Omega')$, and*

$$\|u\|_{W^{2,2}(\Omega')} \leqslant C \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right). \tag{15}$$

where the constant depends on the distance between Ω' and $\partial\Omega$. Furthermore, $\triangle u = f$ almost everywhere in Ω.

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1. Compare with the Poincaré inequality!

Remark 5. The difficulty in proving the theorem lies in the fact that we have to show $u\in W^{2,2}$. Once this is known, the estimate is relatively easy to establish.

1. We first show that

$$\|\nabla u\|^2_{L^2(\Omega')}\leqslant\frac{17}{\delta^2}\|u\|^2_{L^2(\Omega)}+\delta^2\|f\|^2_{L^2(\Omega)}. \tag{16}$$

without any extra assumptions.

Let $\eta(x)$ be a "cut-off" function defined by

$$\eta(x)=\begin{cases}1 & x\in\Omega'\\ 1-\frac{1}{\delta}\operatorname{dist}(x,\Omega') & 0\leqslant\operatorname{dist}(x,\Omega')\leqslant\delta\\ 0 & \operatorname{dist}(x,\Omega')>\delta\end{cases} \tag{17}$$

and set the test function

$$v=\eta^2 u\in W_0^{1,2}(\Omega). \tag{18}$$

Some calculation yields

$$\int_\Omega \eta^2|\nabla u|^2+2\int_\Omega(\eta\nabla u)\cdot(u\nabla\eta)=-\int_\Omega\eta^2 f u. \tag{19}$$

Using Young's inequality

$$|a\,b|\leqslant\frac{\varepsilon}{2}a^2+\frac{1}{2\,\varepsilon}b^2\qquad a,b\in\mathbb{R},\ \varepsilon>0 \tag{20}$$

on the 2nd term on the LHS and on the RHS we have

$$\int_\Omega\eta^2|\nabla u|^2\leqslant\frac{1}{2}\int_\Omega\eta^2|\nabla u|^2+2\int_\Omega u^2|\nabla\eta|^2+\frac{1}{2\,\delta^2}\int\eta^2u^2+\frac{\delta^2}{2}\int\eta^2f^2. \tag{21}$$

Moving the first term on the RHS to the left, we have

$$\int_{\Omega'}|\nabla u|^2\leqslant\left(\frac{16}{\delta^2}+\frac{1}{\delta^2}\right)\int_\Omega u^2+\delta^2\int_\Omega f^2. \tag{22}$$

2. Note that $\int|\triangle u|^2=\int\left|\nabla^2u\right|^2+$ boundary terms. if we assume $u\in W^{3,2}$. Thus using $\triangle u$ as test function we obtain

$$\left\|\nabla^2 u\right\|^2_{L^2(\Omega')}\leqslant\|f\|^2_{L^2(\Omega)}. \tag{23}$$

Proof. Let $\Omega'\subset\subset\Omega''\subset\subset\Omega$, with $\operatorname{dist}(\Omega'',\partial\Omega)\geqslant\delta/4$, $\operatorname{dist}(\Omega',\partial\Omega'')\geqslant\delta/4$.

Now choose $\eta\in C_0^1(\Omega'')$ with $\eta=1$ on Ω' and $|\nabla\eta|\leqslant 8/\delta$, and set

$$v=\eta^2\triangle_i^h u \tag{24}$$

where

$$\triangle_i^h u(x)=\frac{u(x+h\,e_i)-u(x)}{h} \tag{25}$$

Then we have

$$\begin{aligned}\int_{\Omega''}\nabla\left(\triangle_i^h u\right)\cdot\nabla v &= \int_{\Omega''}\triangle_i^h(\nabla u)\cdot\nabla v\\ &= -\int_{\Omega''}\nabla u\cdot\nabla\left(\triangle_i^h v\right)\\ &= \int_{\Omega''} f\,\triangle_i^h v\\ &\leqslant \|f\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega'')}.\end{aligned} \tag{26}$$

Recalling

$$v=\eta^2\triangle_i^h u \tag{27}$$

We have

$$\int_{\Omega''} \nabla\left(\triangle_i^h u\right)\cdot\nabla\left(\eta^2\,\triangle_i^h u\right)\leqslant\|f\|_{L^2(\Omega)}\left\|\nabla\left(\eta^2\,\triangle_i^h u\right)\right\|_{L^2(\Omega'')}. \tag{28}$$

The terms can be expanded to obtain

$$\begin{aligned}\int_{\Omega''}\eta^2\left|\nabla\left(\triangle_i^h u\right)\right|^2 &\leqslant \|f\|_{L^2(\Omega)}\left\|\nabla\left(\eta^2\,\triangle_i^h u\right)\right\|_{L^2(\Omega'')}-2\int_{\Omega''}\left(\eta\,\nabla\triangle_i^h u\right)\cdot\left(\triangle_i^h u\,\nabla\eta\right)\\ &\leqslant 2\,\|f\|_{L^2(\Omega)}^2+\frac{1}{8}\,\left\|\nabla\left(\eta^2\,\triangle_i^h u\right)\right\|_{L^2(\Omega'')}^2+\frac{1}{4}\int_{\Omega''}\left|\eta\,\nabla\triangle_i^h u\right|^2+8\int_{\Omega''}|\nabla\eta|^2\left|\triangle_i^h u\right|^2.\end{aligned} \tag{29}$$

This gives

$$\frac{3}{4}\int_{\Omega''}\left|\eta\,\nabla\triangle_i^h u\right|^2\leqslant 2\,\|f\|_{L^2(\Omega)}^2+\frac{1}{8}\,\left\|\nabla\left(\eta^2\,\triangle_i^h u\right)\right\|_{L^2(\Omega'')}^2+8\int\;|\nabla\eta|^2\left|\triangle_i^h u\right|^2. \tag{30}$$

To proceed further, we need to study the two terms $\frac{1}{8}\,\left\|\nabla\left(\eta^2\,\triangle_i^h u\right)\right\|_{L^2(\Omega'')}^2$ and $8\int\;|\nabla\eta|^2\left|\triangle_i^h u\right|^2$.

- $\frac{1}{8}\,\left\|\nabla\left(\eta^2\,\triangle_i^h u\right)\right\|_{L^2(\Omega'')}^2$. We have

$$\begin{aligned}\left\|\nabla\left(\eta^2\,\triangle_i^h u\right)\right\|_{L^2(\Omega'')}^2 &\leqslant 2\left\|\left(\nabla\left(\eta^2\right)\right)\left|\triangle_i^h u\right|\right\|_{L^2}^2+2\left\|\eta^2\left|\nabla\triangle_i^h u\right|\right\|_{L^2}^2\\ &\leqslant 2\left(\sup\left|\nabla\left(\eta^2\right)\right|\right)\left\|\triangle_i^h u\right\|_{L^2}^2+2\left\|\eta\left|\nabla\triangle_i^h u\right|\right\|_{L^2}^2.\end{aligned} \tag{31}$$

  Where we have used the fact that $\eta^2\leqslant\eta$.

- $8\int\;|\nabla\eta|^2\left|\triangle_i^h u\right|^2$. We have

$$\int\;|\nabla\eta|^2\left|\triangle_i^h u\right|^2\leqslant\left(\sup|\nabla\eta|^2\right)\int\;\left|\triangle_i^h u\right|^2=\left(\sup|\nabla\eta|^2\right)\left\|\triangle_i^h u\right\|_{L^2}^2. \tag{32}$$

Thus we have

$$\frac{1}{2}\int_{\Omega''}\left|\eta\,\nabla\triangle_i^h u\right|^2\leqslant 2\,\|f\|_{L^2(\Omega)}^2+\frac{1}{4}\left(\sup\left|\nabla\left(\eta^2\right)\right|\right)\left\|\triangle_i^h u\right\|_{L^2}^2+\,8\left(\sup|\nabla\eta|^2\right)\left\|\triangle_i^h u\right\|_{L^2}^2. \tag{33}$$

The following lemma then guarantees the existence of $\nabla^2 u$ and also gives the desired estimate.

Lemma. *Let*

$$\triangle_i^h u\equiv\frac{u(x+h\,e_i)-u(x)}{h},\qquad h\neq 0 \tag{34}$$

with e_i being the ith unit vector of $\mathbb{R}^n$. Let $\Omega'\subset\subset\Omega$ and $|h|<\operatorname{dist}(\Omega',\partial\Omega)$. Then

1. *If $u\in L^2(\Omega)$ and there is $K<\infty$ such that*

$$\left\|\triangle_i^h u\right\|_{L^2(\Omega')}\leqslant K \tag{35}$$

   then $u\in W^{1,2}(\Omega')$ and

$$\|\partial_{x_i}u\|_{L^2(\Omega')}\leqslant K. \tag{36}$$

2. *Conversely, if $u\in W^{1,2}(\Omega')$, then $\triangle_i^h u\in L^2(\Omega')$ with*

$$\left\|\triangle_i^h u\right\|_{L^2(\Omega')}\leqslant\|\partial_{x_i}u\|_{L^2(\Omega')}. \tag{37}$$

Proof.

1. We first show that $\triangle_i^h u$ converges as distributions in $\mathcal{D}'(\Omega')$ to the distributional derivative of u. Check

$$\int_{\Omega'}\left(\triangle_i^h u\right)\varphi=-\int u\left(\triangle_i^{-h}\varphi\right)\longrightarrow-\int u\,(\partial_{x_i}\varphi) \tag{38}$$

   by Lebesgue's dominated convergence theorem.

   We have

$$(\partial_{x_i}u)(\varphi)=\lim\int_{\Omega'}\left(\triangle_i^h u\right)\varphi\leqslant\left\|\triangle_i^h u\right\|_{L^2(\Omega')}\|\varphi\|_{L^2}\leqslant K\,\|\varphi\|_{L^2},\qquad\forall\varphi\in C_0^\infty(\Omega'). \tag{39}$$

Now recall that $C_0^\infty(\Omega')$ is dense in $L^2(\Omega')$, $(\partial_{x_i} u)$ can be identified with a bounded linear operator on L^2, which means it can be identified with a function in $L^2(\Omega')$.[2]

2. Since C^∞ is dense in $W^{1,2}$, we only need to consider the case when $u \in C^\infty \cap W^{1,2}$. In this case we have
$$\triangle_i^h u(x) = \frac{1}{h} \int_0^h \partial_{x_i} u(x_1, \ldots, x_{i-1}, x_i + s, x_{i+1}, \ldots, x_n) \, \mathrm{d}s. \tag{40}$$
This gives
$$\left| \triangle_i^h u(x) \right|^2 \leqslant \frac{1}{h} \int_0^h \left| \partial_{x_i} u(x_1, \ldots, x_{i-1}, x_i + s, x_{i+1}, \ldots, x_n) \right|^2 \mathrm{d}s \tag{41}$$
due to Hölder's inequality. Now integrate over Ω' and exchange the order of integration on the RHS we obtain the result. □

With the help of this lemma (part b)) we have
$$\frac{1}{2} \int_{\Omega''} \left| \eta \, \nabla \triangle_i^h u \right|^2 \leqslant 2 \, \|f\|_{L^2(\Omega)}^2 + \frac{1}{4} \left(\sup \left| \nabla \left(\eta^2 \right) \right| \right) \|\partial_{x_i} u\|_{L^2}^2 + 8 \left(\sup |\nabla \eta|^2 \right) \|\partial_{x_i} u\|_{L^2}^2. \tag{42}$$
which is a uniform bound on
$$\left\| \triangle_i^h \nabla u \right\|_{L^2(\Omega'')} \leqslant \int_{\Omega''} \left| \eta \, \nabla \triangle_i^h u \right|^2. \tag{43}$$
Now part a) of the lemma yields $\partial_{x_i} \nabla u \in L^2(\Omega'')$ and also the desired estimate. □

When f has better regularity, we can differentiate the equation first and obtain the following interior regularity result.

Theorem 6. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\triangle u = f$. If $f \in W^{k,2}(\Omega)$, then $u \in W^{k+2,2}(\Omega')$ for any $\Omega' \subset \subset \Omega$, and*
$$\|u\|_{W^{k+2,2}(\Omega')} \leqslant C \left(\|u\|_{L^2(\Omega)} + \|f\|_{W^{k,2}(\Omega)} \right). \tag{44}$$
Here the constant depends on $d, h, \operatorname{dist}(\Omega', \Omega)$.

2.2. Boundary regularity.

We consider the Poisson equation with Dirichlet boundary condition:
$$\triangle u = f \qquad \text{in } \Omega; \qquad u = g \quad \text{on } \partial\Omega \tag{45}$$
where g can be extended to a function on the whole Ω. Our purpose is to establish the following result:

Theorem 7. *Let u be a weak solution with $u - g \in W_0^{1,2}(\Omega)$. If $f \in W^{k,2}(\Omega)$, $g \in W^{k+2,2}(\Omega)$, and Ω be of class C^{k+2}, then*
$$u \in W^{k+2,2}(\Omega), \tag{46}$$
and we have the estimate
$$\|u\|_{W^{k+2,2}(\Omega)} \leqslant C \left(\|f\|_{W^{k,2}(\Omega)} + \|g\|_{W^{k+2,2}(\Omega)} \right). \tag{47}$$
The constant C depends on Ω.

Proof. The proof of this theorem is identical to the proof of a similar theorem for general linear elliptic equations (we will see why soon). As we will not discuss details about the general case, we will only give an outline here. For details see J. Jost **Partial Differential Equations** §8.3.

1. First note that since $g \in W^{k+2,2}(\Omega)$, we can replace u by $u - g$ and reduce the problem to
$$\triangle u = f, \qquad u \in W_0^{1,2}(\Omega). \tag{48}$$

2. Riesz representation theorem.

2. We first establish $W^{1,2}$ bound:

$$\|u\|_{W^{1,2}} \leqslant C\left(\|g\|_{W^{1,2}} + \|f\|_{L^2}\right). \tag{49}$$

To see this, use $v = u - g$ as the test function. We obtain

$$\left|\int \nabla u \cdot \nabla(u-g)\right| = \left|\int f\,(u-g)\right| \tag{50}$$

therefore

$$\int |\nabla u|^2 \leqslant \left|\int \nabla u \cdot \nabla g\right| + \left|\int f\,(u-g)\right| \leqslant \frac{1}{4}\int |\nabla u|^2 + \int |\nabla g|^2 + \frac{1}{\varepsilon}\int f^2 + \varepsilon \int |u-g|^2. \tag{51}$$

Apply Poincaré's inequality to the last term and choosing ε to be small enough, we obtain the desired estimate.

3. For any $\Omega' \subset\subset \Omega$, we can estimate $\int_{\Omega'} \left|\partial_{x_i x_j} u\right|^2 \leqslant C\left(\int u^2 + \int f^2\right) \leqslant C\left(\|g\|_{W^{1,2}} + \|f\|_{L^2}\right)$. Therefore it suffices to establish the desired estimate in a neighborhood of the bondary $\partial\Omega$.

4. We illustrate the basic idea by assuming part of the boundary is in $x_n = 0$. We try to show the $W^{2,2}$ bound for u in a small half-ball $B_R^+ \equiv B_R \cap \{x_n > 0\}$. Note that once this is done, the boundary, which is compact, can be covered by finitely many such balls.

   First note that $\partial_{x_i} u$ is well defined in B_R^+ and belongs to $L^2\left(B_R^+\right)$. Now let η be a cut-off function in $C_0^\infty(B_R)$. For all $j \neq n$, $\triangle_j^{\pm h} u$ is well-defined and we can use the test function $\triangle_j^h\left(\eta^2 \triangle_j^h u\right)$ as we did when proving the interior regularity, and obtain the desired bound for all $\partial_{x_i x_j} u$ except $\partial_{x_n x_n} u$.

   Now notice that the equation implies

$$\partial_{x_n x_n} u = f - \sum_{i=1}^{n-1} \partial_{x_i x_i} u \tag{52}$$

and therefore this term enjoys the same bound as other double derivatives.

5. For general Ω, we need to first cover $\partial\Omega$ by small balls, and then do a change of variable on each of the balls to "straighten" that part of the boundary. After doing this, however, the equation does not have the simple form

$$\triangle u = f \tag{53}$$

anymore and proving the estimate becomes as difficult as proving similar estimates for the general case. □

Remark 8. It turns out that when the boundary is smooth, one can actually extend the regularity to $\bar{\Omega}$.

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^∞, and let $g \in C^\infty(\partial\Omega)$, $f \in C^\infty(\Omega)$. Then the Dirichlet problem*

$$\triangle u = f \quad \text{in } \Omega; \qquad u = g \quad \text{on } \partial\Omega, \tag{54}$$

possesses a unique solution u which is $C^\infty\left(\bar{\Omega}\right)$.

The key to the proof is the embedding $W^{k,p}(\Omega) \subset C^m\left(\bar{\Omega}\right)$ for $0 \leqslant m < k - \frac{d}{p}$. For details, see J. Jost **Partial Differential Equations**, §8.4.

3. L^p regularity.

The regularity theory can be extended to spaces $W^{k,p}$. We will only sketch the results here. For details, see J. Jost **Partial Differential Equations**, Chap. 9.

For $f \in L^p(\Omega)$, we can extend it by 0 to obtain $f \in L^p(\mathbb{R}^n)$. Recall the fundamental solutions:

$$\Gamma(x,y) \equiv \begin{cases} \dfrac{1}{2\pi}\log|x-y| & n=2 \\ \dfrac{1}{n\,(2-n)\,\alpha(n)}\,|x-y|^{2-n} & n \geqslant 3 \end{cases}. \tag{55}$$

We can define the Newton potential

$$w(x) \equiv \int \Gamma(x, y)\, f(y)\, \mathrm{d}y. \tag{56}$$

which solves

$$\triangle u = f \tag{57}$$

almost everywhere.

The basis for the L^p theory is the following Calderon-Zygmund inequality.

Theorem 9. *Let $1 < p < \infty$, $f \in L^p(\Omega)$, and let w be the Newton potential of f. Then $w \in W^{2,p}(\Omega)$, $\triangle w = f$ almost everywhere in Ω, and*

$$\left\| \nabla^2 w \right\|_{L^p(\Omega)} \leqslant C(n, p)\, \| f \|_{L^p(\Omega)}. \tag{58}$$

Using this theorem, we can obtain the following interior regularity result.

Theorem 10. *Let $u \in W^{1,1}(\Omega)$ be a weak solution of $\triangle u = f$, $f \in L^p(\Omega)$, $1 < p < \infty$. Then $u \in W^{2,p}(\Omega')$ for any $\Omega' \subset\subset \Omega$, and*

$$\| u \|_{W^{2,p}(\Omega')} \leqslant C \left(\| u \|_{L^p(\Omega)} + \| f \|_{L^p(\Omega)} \right). \tag{59}$$

Here $C = C(n, p, \Omega', \Omega)$.

Further readings.

- J. Jost, **Partial Differential Equations**, Chap 8, 9.

Exercises.

Exercise 1. Let

$$f(x) = |x|^s, \qquad s \in \mathbb{R}. \tag{60}$$

Consider the domains

$$\Omega = B_R, \quad \Omega' = \mathbb{R}^n \backslash B_R. \tag{61}$$

For which values of p, n, s is $f \in W^{1,p}(\Omega)$ or $W^{1,p}(\Omega')$?

(Optional) How about $W^{k,p}(\Omega)$, $W^{k,p}(\Omega')$?

Exercise 2. Write down a proof of the higher regularity result.

Theorem. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\triangle u = f$. If $f \in W^{k,2}(\Omega)$, then $u \in W^{k+2,2}(\Omega')$ for any $\Omega' \subset\subset \Omega$, and*

$$\| u \|_{W^{k+2,2}(\Omega')} \leqslant C \left(\| u \|_{L^2(\Omega)} + \| f \|_{W^{k,2}(\Omega)} \right). \tag{62}$$

Here the constant depends on $d, h, \operatorname{dist}(\Omega', \Omega)$.

Exercise 3. (Optional) Use the L^2 version of the Calderon-Zygmund inequality:

$$\left\| \nabla^2 w \right\|_{L^2(\Omega)} \leqslant \| f \|_{L^2(\Omega)} \tag{63}$$

where w is the Newton potential of f, to prove the inner regularity estimate on a ball: for any weak solution u of $\triangle u = f$, we have

$$\| u \|_{W^{2,2}(B_r)} \leqslant C \left(\| u \|_{L^2(B_R)} + \| f \|_{L^2(B_R)} \right). \tag{64}$$

Where $r < R$. For simplicity, you do not need to worry about the regularity of u, that is, work as if $u \in C^\infty$.

Hint: Follow these steps.

1. For any $v \in C_0^\infty(B_R)$, $v = \int \Gamma(x, y)\, (\triangle v)(y)\, \mathrm{d}y$. That is, v is the Newton potential of its Laplacian.
2. Use a cut-off function η whose support is in $B_{r'}$ with $r < r' < R$, apply C-Z inequality on $v = \eta u$ to obtain

$$\left\| \nabla^2 u \right\|_{L^2(B_r)} \leqslant C \left(\| u \|_{L^2(B_{r'})} + \| f \|_{L^2(B_{r'})} + \| \nabla u \|_{L^2(B_{r'})} \right). \tag{65}$$

3. Use another cut-off function ξ whose support is in B_R, to obtain

$$\| \nabla u \|_{L^2(B_{r'})} \leqslant C \left(\| u \|_{L^2(B_R)} + \| f \|_{L^2(B_R)} \right). \tag{66}$$