The Dirichlet's Principle

In this lecture we discuss an alternative formulation of the Dirichlet problem for the Laplace equation:

$$\triangle u = 0 \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{1}$$

1. Dirichlet's Principle.

If we multiply the equation by any $v \in C_0^\infty(\Omega)$ and integrate, we have

$$0 = \int (\triangle u)\, v = -\int \nabla u \cdot \nabla v. \tag{2}$$

As a consequence, we have

$$\int |\nabla(u+v)|^2 = \int |\nabla u|^2 + \int |\nabla v|^2 \geqslant \int |\nabla u|^2. \tag{3}$$

In other words, u is the minimizer of the function

$$D(u) \equiv \int_\Omega |\nabla u|^2 \, \mathrm{d}x. \tag{4}$$

Conversely, if u is a minimizer, then for any $v \in C_0^\infty$, and $t > 0$, we have

$$\int |\nabla(u + t\, v)|^2 \geqslant \int |\nabla u|^2 \iff t^2 \int |\nabla v|^2 - 2\, t \int \nabla u \cdot \nabla v \geqslant 0 \tag{5}$$

which implies

$$\int (\triangle u)\, v = -\int \nabla u \cdot \nabla v = 0 \tag{6}$$

by taking $t \searrow 0$ and consequently

$$\triangle u = 0 \tag{7}$$

when $u \in C^2$.

From the above discussion we conclude the following Dirichlet principle.

Dirichlet principle. u solves

$$\triangle u = 0 \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{8}$$

if and only if u minimizes

$$D(u) \equiv \int_\Omega |\nabla u|^2 \, \mathrm{d}x. \tag{9}$$

A moment's inspection reveals that the principle cannot be automatically true without specifying the class of functions u should belong to: $D(u)$ is well-defined when $u \in C^1$ but u needs to be in C^2 to satisfy the Laplace's equation. And furthermore, it is not clear yet why $D(u)$ should have a minimizer.

We first establish

Theorem 1. *$D(u)$ has a minimizer u satisfying $\int |\nabla u|^2 < \infty$, that is $\nabla u \in L^2$.*

Proof. Let u_n be a minimizing sequence, that is

$$\lim_{n\to\infty} D(u_n) = \inf D(u). \tag{10}$$

Then one calculates

$$\begin{aligned} D(u_n - u_m) &= \int |\nabla u_n - \nabla u_m|^2 \\ &= \int |\nabla u_n|^2 - 2\,\nabla u_n \cdot \nabla u_m + |\nabla u_m|^2 \\ &= 2\int |\nabla u_n|^2 + 2\int |\nabla u_m|^2 - \int |\nabla u_n + \nabla u_m|^2 \\ &= 2\,D(u_n) + 2\,D(u_m) - 4\,D\left(\frac{u_n + u_m}{2}\right). \end{aligned} \tag{11}$$

Since

$$4\,D\left(\frac{u_n + u_m}{2}\right) \geqslant 4 \inf D(u) = \lim\,[2\,D(u_n) + 2\,D(u_m)], \tag{12}$$

we see that

$$D(u_n - u_m) \to 0 \qquad n, m \to \infty \tag{13}$$

or equivalently $\{\nabla u_n\}$ is a Cauchy sequence in the space L^2 of all square integrable functions. Thus there is a limit function $w = \lim \nabla u_n$ which is square integrable.

It turns out that

1. $w = \nabla u$ for some function u in the sense of distributions.
2. $D(u) \leqslant \lim D(u_n)$ which means u is a minimizer. □

From the above theorem we see that only the existence of ∇u (as a square integrable function) is guaranteed. Therefore the Dirichlet principle only makes sense when we re-define the Laplace equation to its weak formulation:

$$\int \nabla u \cdot \nabla v = 0 \qquad \forall v \in C_0^\infty(\Omega), \qquad u = g \quad \text{on } \partial\Omega. \tag{14}$$

2. The Sobolev space $W^{1,2}(\Omega)$.

Definition 2. *The Sobolev space $W^{1,2}(\Omega)$ is defined as the space of those $u \in L^2(\Omega)$ whose distributional derivatives $\partial_{x_i} u$ also belong to $L^2(\Omega)$.*

Proposition 3.

i. $W^{1,2}(\Omega)$ becomes a Hilbert space after we define the inner product

$$(u, v)_{W^{1,2}(\Omega)} \equiv \int_\Omega u\,v + \sum_{i=1}^n \int_\Omega \partial_{x_i} u\, \partial_{x_i} v. \tag{15}$$

The induced norm is

$$\|u\|_{W^{1,2}(\Omega)} \equiv (u, u)_{W^{1,2}(\Omega)}^{1/2}. \tag{16}$$

ii. $C^\infty(\Omega)$ is dense in $W^{1,2}(\Omega)$.

Proof. See J. Jost **Partial Differential Equations**, §7.2. □

Example 4.

1. $u(x) = |x| \in W^{1,2}(-1, 1)$;
2. $u(x) = \begin{cases} 1 & 0 \leqslant x < 1 \\ 0 & -1 < x < 0 \end{cases} \notin W^{1,2}(-1, 1)$.

Definition 5. *The closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$ is denoted $W_0^{1,2}(\Omega)$.*

Example 6.

1. $u(x) = 1 - |x| \in W_0^{1,2}(-1,1)$.
2. $u(x) \equiv 1 \notin W_0^{1,2}(-1,1)$.[1]

Remark 7. Intuitively, $W_0^{1,2}(\Omega)$ are those functions in $W^{1,2}(\Omega)$ which are 0 on the boundary.

The following properties are important for studying PDEs. We will omit their proofs, details can be found in J. Jost **Partial Differential Equations**, §7.2.

Lemma 8.

- *For $u \in W^{1,2}(\Omega)$, $f \in C^1(\Omega)$, suppose*
$$\sup_{y \in \mathbb{R}} |f'(y)| < \infty. \tag{19}$$
Then $f \circ u \in W^{1,2}(\Omega)$ and $D(f \circ u) = f'(u)\, Du$.
- *The above is still true when $f \in \mathrm{Lip}(\Omega)$.* [2] *In particular, if $u \in W^{1,2}(\Omega)$, so is $|u|$, and*
$$D|u| = (\operatorname{sign} u)\, Du. \tag{20}$$

Now recall the minimization problem in the Dirichlet principle:

$$\min \int |\nabla u|^2, \qquad u = g \quad \text{on } \partial\Omega. \tag{21}$$

We would like to rigorously specify over which set the minimization is taking place. This set is exactly the space $W^{1,2}$. Thus we would like to minimize over all functions in $W^{1,2}$ with boundary value g. Recall that if u_n is a minimizing sequence, then ∇u_n is a Cauchy sequence in L^2. If furthermore u_n is a Cauchy sequence in L^2 too, we know that the sequence is a Cauchy sequence in $W^{1,2}$.

The following Poincaré inequality guarantees that u_n is a Cauchy sequence in L^2.

Lemma 9. *There is a constant C, depending on the bounded set Ω only, such that for all $u \in W_0^{1,2}(\Omega)$, we have*

$$\|u\|_{L^2(\Omega)} \leqslant C\ \|\nabla u\|_{L^2(\Omega)}. \tag{22}$$

Proof. We prove by contradiction. Assume that there are u_k such that

$$\|u_k\|_{L^2(\Omega)} > k\ \|\nabla u_k\|_{L^2(\Omega)}. \tag{23}$$

Rescaling, we can set $\|u_k\|_{L^2} = 1$. Thus $\nabla u_k \to 0$ in L^2. By the compactness theorem of Rellich[3], there is a subsequence u_{k_j} which convergens in L^2. Thus $\{u_{k_j}\}$ converges in $W_0^{1,2}$ to some limit function u satisfying

$$\|u\|_{L^2} = 1, \qquad \nabla u = 0, \qquad u \in W_0^{1,2} \tag{24}$$

where the contradiction is obvious. □

1. To see this, we assume the contrary, that is there are $u_n \in C_0^\infty(-1,1)$ such that $u_n \to u \equiv 1$ in $W^{1,2}$. This means

$$\int_{-1}^{1} (u_n - 1)^2\, \mathrm{d}x \longrightarrow 0, \qquad \int_{-1}^{1} (u_n')^2 \longrightarrow 0. \tag{17}$$

But the latter implies

$$|u_n(x)| \leqslant \left| \int_{-1}^{x} u_n' \right| \leqslant \int_{-1}^{1} |u_n'|\, \mathrm{d}x \leqslant \sqrt{2} \left(\int_{-1}^{1} (u_n')^2\, \mathrm{d}x \right)^2 \longrightarrow 0 \tag{18}$$

for any $x \in (-1,1)$ and furthermore the convergence is uniform in x. Contradiction.

2. *The idea is to approximate f by C^1 functions, and use Lebesgue's dominated convergence Theorem.*

3. **Theorem.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $\{u_n\}$ be a bounded sequence in $W_0^{1,2}(\Omega)$, then there is a subsequence which converges strongly in $L^2(\Omega)$.*

Remark 10. The following argument yields an explicit C when Ω is contained in a box of side R (denote the box by C_R):

Extend u by 0 outside Ω we obtain a $W^{1,2}$ function, still denoted u, defined on the box. Without loss of generality we assume the box is $0 \leqslant x_i \leqslant R$. Integrating from $x_n = 0$ we have

$$u(x_1, \ldots, x_n) = \int_0^{x_n} \partial_{x_n} u(x_1, \ldots, x_{n-1}, t)\,\mathrm{d}t. \tag{25}$$

Now we have

$$\begin{aligned}
\int |u|^2 &\leqslant \int \left(|u(x_1, \ldots, x_n)| \int_0^{x_n} |\nabla u|\,\mathrm{d}t \right) \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&\leqslant \int \left(|u(x_1, \ldots, x_n)| \int_0^{R} |\nabla u|\,\mathrm{d}t \right) \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&= \int\int_0^R |u(x_1, \ldots, x_n)|\ |\nabla u(x_1, \ldots, t)|\,\mathrm{d}t\,\mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&\leqslant \left(\int\int_0^R |u(x_1, \ldots, x_n)|^2 \right)^{1/2} \left(\int\int |\nabla u(x_1, \ldots, t)|\,\mathrm{d}t\,\mathrm{d}x_1 \cdots \mathrm{d}x_n \right)^{1/2} \\
&\leqslant R^{1/2} \left(\int |u|^2 \right)^{1/2} R^{1/2} \left(\int |\nabla u|^2 \right)^{1/2}.
\end{aligned} \tag{26}$$

and thus obtaining

$$\|u\|_{L^2(\Omega)} \leqslant R\, \|\nabla u\|_{L^2(\Omega)}. \tag{27}$$

A more refined (and more general, as it can be applied to unbounded regions) estimate has the constant

$$C = \left(\frac{|\Omega|}{\alpha(n)} \right)^{1/n} \tag{28}$$

where $|\Omega|$ is the volume of Ω and $\alpha(n)$ is the volume of the n-dimensional unit ball. See the proof of Theorem 7.2.2 in J. Jost **Partial Differential Equations**.

3. Weak formulation.

From the above discussion we see that the minimizer of the Dirichlet functional is in $W^{1,2}(\Omega)$. Now we are ready to give the definition of a solution $u \in W^{1,2}(\Omega)$:

Definition 11. $u \in W^{1,2}(\Omega)$ *is a weak solution of the Laplace equation*

$$\triangle u = 0, \quad \text{in } \Omega, \qquad u = g, \quad \text{on } \partial\Omega \tag{29}$$

if

$$\int \nabla u \cdot \nabla v = 0 \quad \forall v \in W_0^{1,2}(\Omega); \qquad u - g \in W_0^{1,2}(\Omega). \tag{30}$$

Remark 12.

1. This definition requires that the boundary value g can be extended to a function in $W^{1,2}(\Omega)$. This can indeed be done. See e.g. R. A. Adams **Sobolev Spaces**.

2. Since C_0^∞ is dense in $W_0^{1,2}(\Omega)$ (by definition!), we can also use $\forall v \in C_0^\infty$ in the definition. The current definition is however more convenient. For example, when the solution u exists, the non-smooth function $\max\{0, u - k\} \in W_0^{1,2}(\Omega)$ (if k is bigger than the boundary values) can be used as test functions. [4]

4. This choice of test functions is used in the so-called De Giorgi method, which obtains L^∞ bound from energy bound.

4. Poisson equation.

The above discussions can be applied to the Poisson equation

$$\triangle u = f, \quad \text{in } \Omega, \qquad u = g, \quad \text{on } \partial\Omega \tag{31}$$

with little modification. In this case, the definition for weak solutions is

Definition 13. $u \in W^{1,2}(\Omega)$ *is a weak solution of the Poisson equation*

$$\triangle u = f, \quad \textit{in } \Omega, \qquad u = g, \quad \textit{on } \partial\Omega \tag{32}$$

if

$$\int \nabla u \cdot \nabla v + \int f v = 0 \quad \forall v \in W_0^{1,2}(\Omega); \qquad u - g \in W_0^{1,2}(\Omega). \tag{33}$$

The weak formulation is advantageous in getting quick estimates. For example, when $g = 0$, we have

$$\|u\|_{W^{1,2}} \leqslant C \, \|f\|_{L^2} \tag{34}$$

for some constant C.

To see this, note that when $g = 0$, $u \in W_0^{1,2}$ can be used as a test function, which gives

$$\int |\nabla u|^2 = -\int f u \leqslant \|f\|_{L^2} \|u\|_{L^2}. \tag{35}$$

Applying Poincaré inequality gives the desired estimate.

5. Introduction to the direct method.

The direct method shows the existence/uniqueness of the solution of PDEs by studying its variational formulation. We sketch this approach by studying the Poisson equation with zero boundary condition:

$$\triangle u = f, \qquad u \in W_0^{1,2}(\Omega). \tag{36}$$

We know that any weak solution to this problem is a minimizer of the functional

$$D(u) = \int_\Omega |\nabla u|^2 + \int f u. \tag{37}$$

We would like to show that the minimizer exists and is unique. An outline of the argument is the following. Assume $f \in L^2$.

1. Writing

   $$D(u) \geqslant \int_\Omega |\nabla u|^2 - \varepsilon \int u^2 - \frac{1}{4\varepsilon} \int f^2 \tag{38}$$

   and recalling the Poincaré's inequality, we see that $D(u)$ has finite infimum.

2. Let u_n be such that $D(u_n) \searrow \inf_{u \in W_0^{1,2}} D(u)$. We claim that there is a subsequence converging to some limit $u_\infty \in W_0^{1,2}$.[5] To see this, note that a uniform bound on $D(u_n)$ implies a uniform bound on $\int |\nabla u_n|^2$, since

   $$D(u) \geqslant \|\nabla u\|_{L^2}^2 - \|u\|_{L^2} \|f\|_{L^2} \geqslant \|\nabla u\|_{L^2}^2 - C \, \|\nabla u\|_{L^2} = \left(\|\nabla u\|_{L^2} - C\right) \|\nabla u\|_{L^2}. \tag{39}$$

   by Hölder's inequality and Poincaré's inequality.

3. Uniform boundedness of $\|\nabla u_n\|_{L^2}$ implies that u_n is uniformly bounded in $W_0^{1,2}$ and thus has a weakly[6] converging subsequence, still denoted by u_n. We denote the limit by u_∞.

5. This actually cannot be guaranteed now.

6. The weak convergence is in $W^{1,2}$. Recall that a sequence $\{u_n\}$ in a Hilbert space H is weakly convergent with weak limit $u_\infty \in H$ if $(u_n, v) \longrightarrow (u_\infty, v)$ for any $v \in H$.

Furthermore, using Rellich's theorem, we see that when u_n converges to u_∞ weakly in $W^{1,2}$, we can find a subsequence, still denoted u_n, converging to u_∞ strongly in L^2, at the same time ∇u_n converges to ∇u_∞ weakly in L^2.

4. The convexity of the functional $D(u)$ then guarantees that

$$D(u_\infty) \leqslant \liminf_{n \nearrow \infty} D(u_n) = \inf_{u \in W_0^{1,2}} D(u) \tag{40}$$

which means u_∞ is a minimizer.

5. The convexity of $D(u)$ also guarantees the uniqueness of the minimizer.

The last few steps in general involve much technicality.[7] Interested readers can refer to the book by B. Dacorogna for details.

Remark 14. This approach easily generalizes to certain nonlinear equations of the form:

$$-\nabla \cdot \left(\frac{\partial F}{\partial p}(x, u, \nabla u) \right) + \frac{\partial F}{\partial u}(x, u, \nabla u) = 0. \tag{43}$$

where $F(x, u, p)$ is smooth and convex in p, with certain growth condition at infinity. The key observation is that this equation is the condition for minimizers of the functional

$$D(u) = \int_\Omega F(x, u, \nabla u) \, \mathrm{d}x. \tag{44}$$

The books by B. Dacorogna and L. C. Evans are good texts for direct methods in variational problems.

Further readings.

- J. Jost, **Partial Differential Equations**, Chap. 7.
- B. Dacorogna, **Direct Methods in the Calculus of Variations**.
- L. C. Evans, **Weak Convergence Methods for Nonlinear Partial Differential Equations**.

Exercises.

Exercise 1. Consider the functional

$$D_p(u) \equiv \int_\Omega |\nabla u|^p \, \mathrm{d}x, \qquad u = g \quad \text{on } \partial\Omega. \tag{45}$$

where $1 < p < \infty$. What equation should its minimizer satisfy?[8]

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7. In our case much technicality is not involved. The only thing required is some familiarity with weak convergence in Hilbert spaces. For example, note that when $u_n \rightarrow u_\infty$ weakly,

$$(u_n, u_n) - (u_\infty, u_\infty) = \lim \left[(u_n - u_\infty, u_n - u_\infty) \right] \geqslant 0. \tag{41}$$

One can even show similarly that the convergence is in fact strong.
Furthermore if u_∞ and v_∞ are both minimizers of the norm, we have

$$0 = (u_\infty, u_\infty) - (v_\infty, v_\infty) = (u_\infty - v_\infty, u_\infty - v_\infty) \tag{42}$$

since $(v_\infty, u_\infty - v_\infty)$ must vanish due to the fact that v_∞ is a minimizer (local minimizer is enough).

8. The resulting operator is called the "p-Laplacian".