Math 497 R1 Winter 2018 Navier-Stokes Regularity

Lecture 1: Sobolev Spaces and Newtonian Potentials

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Based on §1.1–1.2 of [1]. Some fine properties of Sobolev spaces, and basics of Newtonian potentials.

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1. Sobolev Spaces

1.1. Lebesgue spaces

LEMMA 1. (LEMMA 1.1 OF [1]) *Let $1 \leqslant p < \infty$, then $L_p(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $L_p(\Omega)$.*

Proof.

1. We have $\Omega = \cup B_n$ where B_n has center $x_n \in \Omega$ rational and radius $r_n = \frac{1}{2}\operatorname{dist}(x_n, \Omega^c)$.
2. We also have Ω can be approximated from within by compact sets C_m such that the measure of the difference $\Omega - C_m$ goes to zero.
3. Each C_m can be covered by finitely many B_n's. Denote the union of these balls by Ω_m. We now have $\operatorname{dist}(\Omega_m, \Omega^c) > 0$ and $\|u\|_{L_p(\Omega - \Omega_m)} \longrightarrow 0$.
4. Now mollify $u\,\chi_{\Omega_m}$ to get $u_m \in C_0^\infty(\Omega)$. □

Remark 2.

i. Note that Ω is only required to be a domain (open connected set). No regularity is needed for $\partial\Omega$.

ii. It is obvious that L_∞ is not the completion of C_0^∞.

QUESTION 3. *What about Lorentz spaces?*

1.2. Sobolev spaces

DEFINITION 4. (SOBOLEV SPACES)

- $W_s^k(\Omega)$. *L_s integrability of weak derivatives.*

$$\|u\|_{W_s^k(\Omega)} = \sum_{i=0}^{k} \|\nabla^i u\|_{s,\Omega}. \tag{1}$$

- $\mathring{W}_s^k(\Omega)$. *W_s^k completion of $C_0^\infty(\Omega)$.*
- $L_s^k(\Omega)$. *L_s integrability of $\nabla^k u$.*

$$\|u\|_{L_s^k(\Omega)} = \|\nabla^k u\|_{s,\Omega}. \tag{2}$$

- $\mathring{L}_s^k(\Omega)$. *L_s^k completion of $C_0^\infty(\Omega)$ in the following sense. $[u] \in \mathring{L}_s^k(\Omega)$ is an equivalence class of functions satisfying*
  - *a)* $\forall v, w \in [u]$, $\nabla^k(v - w) = 0$, *and*
  - *b)* $\exists u_0 \in [u]$ *such that there exists* $u_m \in C_0^\infty(\Omega)$ *with* $\|\nabla^k(u_m - u_0)\|_{s,\Omega} \longrightarrow 0$.

It turns out that integrability of derivative implies local integrability of the function.

THEOREM 5. (THEOREM 1.1) $u \in L_s^k(\Omega) \Longrightarrow u \in L_{s,\mathrm{loc}}(\Omega)$.

Proof.

It suffices to prove for $k = 1$. Once this is done, the general case follows easily from induction. By assumption, for any $\varphi \in C_0^\infty(\Omega)$ we have $\langle u, \nabla\varphi\rangle = -\int_\Omega g\,\varphi$ for some $g \in L_s(\Omega)$. We need to prove, for any $\Omega_0 \Subset \Omega$, $u \in L_s(\Omega_0)$. The main difficulty lies in construction of the function u which is well-defined in the whole Ω.

Now fix one such Ω_0. Let $0<\varepsilon<\operatorname{dist}(\Omega_0,\,\Omega^c)$. Let u_ε be the standard mollification of the distribution u.

i. We first show that $u_\varepsilon \in L_\infty(\Omega_0)$ for each ε. To see this, define $l\colon L_1(\Omega_0)\mapsto \mathbb{R}$ by
$$l(\psi):=\langle u,\psi_\varepsilon\rangle. \tag{3}$$
As $u\in\mathcal{D}'(\Omega)$, there is $m\in\mathbb{N}\cup\{0\}$ such that $|l(\psi)|\leqslant C\,\|\psi_\varepsilon\|_{C^m(\Omega)}\leqslant C(\varepsilon)\,\|\psi\|_{L^1(\Omega_0)}$. Thus by Riesz representation theorem we have $u_\varepsilon\in L_\infty(\Omega_0)$.

ii. Now we easily see that $g_\varepsilon=\nabla u_\varepsilon$ in Ω_0.

iii. Next let $\bar{u}_{0,\varepsilon}:=u_\varepsilon-[u_\varepsilon]_{\Omega_0}$ where $[u_\varepsilon]_{\Omega_0}$ is the average of u_ε in Ω_0. By Poincare's inequality we have $\bar{u}_{0,\varepsilon}\longrightarrow u_0$ in $L_s(\Omega_0)$. ??? Naturally $g=\nabla u_0$ in Ω_0.

iv. Now we need to construct u, defined on Ω, such that $u|_{\Omega_0}=u_0$. Let $\Omega_1\Supset\Omega_0$ and $\Subset\Omega$. Repeating the above we have $u_1\in L_s(\Omega_1)$ such that $g=\nabla u_1$ in Ω_1. As $\nabla(u_1-u_0)=0$ in Ω_0, $u_1-u_0=C_0$, a constant, in Ω_0. We re-define $u_1-C_0\in L_s(\Omega_1)$[1] as the new u_1. This can be repeated for a nested sequence of sets $\Omega_0\Subset\Omega_1\Subset\Omega_2\Subset\cdots\Subset\Omega$. Note that in each Ω_m, we have $u_m=u_{m+1}=u_{m+2}=\cdots$. Thus convergence is not an issue.

Finally, any other v with $\nabla v=g$ is just a constant away from the u just constructed. □

Remark 6. It is clear that we cannot expect $u\in L_s(\Omega)$. For example let $\Omega=\mathbb{R}$ and $s=\infty$, and $u=|x|$.

Corollary 7. *If $u_m\in C_0^\infty(\Omega)$ is Cauchy in L_s^k, then there is $u\in L_s^k(\Omega)$ such that $\|u_m-u\|_{L_s^k(\Omega)}\longrightarrow 0$.*

Remark 8. The proof is similar to that of Theorem 5. Also note that Ω does not need to be bounded here.

Corollary 9. (Proposition 1.2) *For bounded domain Ω, $\mathring{L}_s^k(\Omega)=\mathring{W}_s^k(\Omega)$.*

Proof. Clearly $\mathring{W}_s^k(\Omega)\subseteq\mathring{L}_s^k(\Omega)$. For the other direction, we need to show that for every $[u]\in\mathring{L}_s^k(\Omega)$, there must exist a $v\in[u]$ such that $v\in\mathring{W}_s^k(\Omega)$.

Let $[u]\in\mathring{L}_s^k(\Omega)$. By definition there is $w\in L_{s,\mathrm{loc}}(\Omega)$ such that

i. For every other $w'\in[u]$, $\nabla^k(w-w')=0$;

ii. There is a sequence $w_m\in C_0^\infty(\Omega)$ such that $\nabla^k w_m\longrightarrow\nabla^k w$ in $L_s(\Omega)$.

As Ω is bounded, we have Friedrichs inequality:
$$\|w_m-w_{m'}\|_{L_s(\Omega)}\leqslant c\,\|\nabla^k w_m-\nabla^k w_{m'}\|_{L_s(\Omega)} \tag{4}$$
which can be obtained through repeated application of Poincaré. Thus w_m converges in L_s, to some function v. It is clear that $v\in[u]$. □

Remark 10. When $n\geqslant 3$, thanks to the Gagliardo-Nirenberg inequality $\|u\|_{p,\Omega}\leqslant c(n)\,\|\nabla u\|_{2,\Omega}$ where $p=\frac{2n}{n-2}$, we see that $\mathring{L}_2^1(\Omega)\subset L_p(\Omega)$, in the sense that for every $[u]\in\mathring{L}_2^1(\Omega)$, there is a $w\in[u]$ such that $w\in L_p$.

When $n=2$ and $\Omega=\mathbb{R}_+^2$, we can still select a "good representative" for every $[u]\in\mathring{L}_2^1(\mathbb{R}_+^2)$ by the criterion $\|v\|_{L_2(\Pi)}<\infty$ where $\Pi=\mathbb{R}\times(0,1)$. $\Longrightarrow$Any Ω with good boundary would be OK?

[1] Ω_1 is bounded.

2. NEWTONIAN POTENTIALS

Recall the fundamental solutions

$$E(x) = \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x|} & n = 2 \\ \frac{1}{\omega_n n (n-2)} \frac{1}{|x|^{n-2}} & n \geqslant 3 \end{cases}. \tag{5}$$

We can define the Newtonian potential of a function f:

$$u = E * f. \tag{6}$$

PROPOSITION 11. (PROPOSITION 2.3) *Let $f \in L_p(\mathbb{R}^n)$ with $1 < p < \infty$ and $u = E * f$. Then $-\triangle u = f$ in $\mathbb{R}^n$, and $u \in \mathring{L}_p^2(\mathbb{R}^n)$. Furthermore $\int_{\mathbb{R}^n} |\nabla^2 u|^p \, \mathrm{d}x \leqslant c(n,p) \int_{\mathbb{R}^n} |f|^p \, \mathrm{d}x$.*

Proof. Note that $\int_{\mathbb{R}^n} |\nabla^2 u|^p \, \mathrm{d}x \leqslant c(n,p) \int_{\mathbb{R}^n} |f|^p \, \mathrm{d}x$ follows from the theory of singular integral operators.

Next notice that for $f \in C_0^\infty$, $-\triangle u = f$ holds by direct calculation. The general situation now follows from the above estimate and approximation argument.

Thus all we need to prove is the existence of $u_m \in C_0^\infty$ such that $\|\nabla^2 (u_m - u)\|_{L_2(\mathbb{R}^n)} \longrightarrow 0$. Let $f_m \in C_0^\infty(\mathbb{R}^n)$, $f_m \longrightarrow f$ in $L_p(\mathbb{R}^n)$. Define $v_m = E * f_m$. As f_m has compact support, we have

$$|\nabla^i v_m| \leqslant \frac{c(m,i)}{|x|^{n-2+i}}, \qquad x \in \mathbb{R}^n. \tag{7}$$

Now consider $R > 0$ and let φ_R be the standard cut-off function. We calculate

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla^2 (\varphi_R v_m - v_m)|^p \, \mathrm{d}x &\leqslant c \left[\int_{B_R^c} |\nabla^2 v_m|^p + \frac{1}{R^p} \int_{B_{2R} \setminus B_R} |\nabla v_m|^p + \frac{1}{R^{2p}} \int_{B_{2R} \setminus B_R} |\nabla^2 v_m|^p \right] \\ &\leqslant c \int_{B_R^c} |\nabla^2 v_m|^p + C(m) \, R^{n(1-p)}. \end{aligned} \tag{8}$$

Thus for each m, we take R_m such that $\int_{\mathbb{R}^n} |\nabla^2 (\varphi_{R_m} v_m - v_m)|^p \, \mathrm{d}x < \frac{1}{m}$. Defining $u_m = \varphi_{R_m} v_m$ finishes the proof. $\square$

Bibliography

[1] Gregory Seregin. *Lecture Notes on Regularity Theory for the Navier-Stokes Equations*. World Scientific, 2015.