Paradigm Counting Problems

We solve paradigm problems using the method of generating functions.

1. The method of generating functions

- Problem to solve: A family of counting problems with a parameter $n = 0, 1, 2, \ldots$ (for example, coloring n balls). Let the answers be $a_0, a_1, \ldots$
- Relate a "generating function", that is a function that "generates" all the answers, to the numbers:
  - Ordinary generating function:
$$A(x) := \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{1}$$
  - Exponential generating function:
$$E(x) := \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = a_0 + a_1 x + \frac{a_2}{2} x^2 + \frac{a_3}{6} x^3 + \cdots. \tag{2}$$
- Analyze the problem to
  - Decide which generating function form is more convenient: $A(x)$ or $E(x)$.
  - Find the generating function
    - directly, or
    - through obtaining an algebraic or differential equation for the function, and then solve the equation.
- Obtain the answer through obtaining the Taylor expansion of the generating function.

2. Occupancy problems through generating functions

Basic idea: The generating function is a product of m factors where m is the number of boxes, with each factor representing all possibilities for a box. More specifically, if $i_1, i_2, \ldots$, are the numbers of balls allowed in the ith box, then the terms in the ith factor are $x^{i_1}, x^{i_2}, \ldots$.

- This basic idea applies well to the cases with distinct boxes.
- Extra work is needed when the boxes are identical.

2.1. Balls identical, boxes different

- Ordinary generating functions.

Example 1. Find the generating function for the number of different ways putting n identical balls into 4 different boxes where the first box cannot be empty, the number of balls in the second box is a multiple of 3, the third box has at least 5 balls, and the fourth box has at most 3 balls.

Solution. We have

$$A(x) = (x + x^2 + \cdots)(1 + x^3 + x^6 + \cdots)(x^5 + x^6 + \cdots)(1 + x + x^2 + x^3). \tag{3}$$

Exercise 1. Find a_{100}.

2.2. Balls different, boxes different

- Exponential generating functions.

Example 2. Find the generating function for the number of different ways putting n different balls into 4 different boxes where the first box cannot be empty, the number of balls in the second box is a multiple of 3, the third box has at least 5 balls, and the fourth box has at most 3 balls.

Solution. We have

$$A(x) = \left(x + \frac{x^2}{2!} + \cdots \right) \left(1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots \right) \left(\frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \right) \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \right). \tag{4}$$

2.3. Balls different, boxes identical

- In general, extra work is needed to apply the method of generating functions to problems with identical boxes.
- Special cases that are relatively easy to do:
  - Putting n different balls into m identical boxes;
  - Putting n different balls into m identical boxes, and the boxes are not allowed to be empty.

2.4. Balls identical, boxes identical

- Identify the problem with a general integer solution problem.

3. Integer solution problems

3.1. Simple problems

- The problem:

$$x_1 + \cdots + x_m = n, \qquad a_i < (\leqslant) x_i < (\leqslant) b_i. \tag{5}$$

- Essentially the same as (take $\leqslant, \leqslant$ as example): Putting n identical balls into m different boxes, with the number of balls in the ith box no less than a_i and no more than b_i.
- Solve by generating function: (still use $\leqslant, \leqslant$)

$$A(x) = (x^{a_1} + \cdots + x^{b_1}) (x^{a_2} + \cdots + x^{b_2}) \cdots (x^{a_m} + \cdots + x^{b_m}). \tag{6}$$

Example 3. Find the number of solutions to

$$x_1 + x_2 + x_3 + x_4 = 20, \qquad -1 \leqslant x_1 \leqslant 7, 2 \leqslant x_2 \leqslant 15, 4 < x_3 < 19, 5 \leqslant x_4 \tag{7}$$

using generating function.

Solution. The answer is given by the coefficient of x^{20} in the expansion of

$$A(x) := (x^{-1} + \cdots + x^7) (x^2 + \cdots + x^{15}) (x^5 + \cdots + x^{18}) (x^5 + \cdots) \tag{8}$$

We calculate

$$\begin{aligned} A(x) &= x^{-1} (1 + \cdots + x^8) x^2 (1 + \cdots + x^{13}) x^5 (1 + \cdots + x^{13}) x^5 (1 + x + \cdots) \\ &= x^{11} \frac{1 - x^9}{1 - x} \frac{1 - x^{14}}{1 - x} \frac{1 - x^{14}}{1 - x} \frac{1}{1 - x} \\ &= x^{11} (1 - x^9) (1 - x^{14})^2 \frac{1}{(1 - x)^4} \\ &= x^{11} (1 - x^9) (1 - x^{14})^2 \left[\sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{3!} x^n \right] \\ &= x^{11} (1 - x^{14})^2 \left[\sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{3!} x^n \right] \\ &\quad - x^{20} (1 - x^{14})^2 \left[\sum_{n=0}^{\infty} \frac{(n+3)(n+2)(n+1)}{3!} x^n \right]. \end{aligned} \tag{9}$$

The coefficient of x^{20} can be now easily found as

$$\frac{(9+3)(9+2)(9+1)}{3!} - 1 = 219. \tag{10}$$

3.2. General problems

- The problem:

$$c_1 x_1 + \cdots + c_m x_m = n, \qquad a_i < (\leqslant) x_i < (\leqslant) b_i. \tag{11}$$

- Generating function approach:
  - Equivalent problem:

$$y_1 + \cdots + y_m = n, \qquad c_i a_i < (\leqslant) y_i < (\leqslant) c_i b_i. \tag{12}$$

  - Essentially the same as (for $\leqslant, \leqslant$): Putting n identical balls into m different boxes, with the number of balls in the ith box satisfying:
    1. is a multiple of c_i;
    2. no less than $c_i a_i$;
    3. no more than $c_i b_i$.

4. Coloring problems

4.1. Simple coloring problems

- Coloring n different balls with m colors.
  Same as putting n different balls into m different boxes.
- Coloring n identical balls with m colors.
  Same as putting n identical balls into m different boxes.

4.2. Coloring problems with symmetry (Polya's theory)

- Coloring n balls with m colors, where the n balls are "in between" being all different or all identical. More specifically,
  - when all the balls are different, the symmetry group $G = \{i\}$.
  - when all the balls are identical, the symmetry group $G = P_n$, the group of all permutations of $\{1, 2, \ldots, n\}$.
- Polya's theory.

$$\text{Ans} = \frac{1}{|G|} \sum_{g \in G} m^{c(g)} \tag{13}$$

  where $c(g)$ is the number of cycles of the permutation g.

  Example 4. In how many ways can a $2 \times 2 \times 2$ cube be constructed from eight $1 \times 1 \times 1$ cubes if an unlimited number of red, white, and blue cubes are available?

  Solution. We see that this problem is equivalent to coloring the eight vertices of a cube with three colors. Polya's theory then applies. For the 24 elements of the symmetry group G of the cube we have

  - i: $c(i) = 8$;
  - 6 rotations of 90 and 270 degrees around lines passing the centers of opposite faces. $c(g) = 2$.
  - 3 rotations of 180 degrees around lines passing the centers of opposite faces. $c(g) = 4$.
  - 8 rotations of 120 and 240 degrees around the long diagonals. $c(g) = 4$.
  - 6 rotations of 180 degrees around the lines passing the middle points of opposite edges. $c(g) = 4$.

  Thus the answer is given by

$$\frac{3^8 + 6 \times 3^2 + 3 \times 3^4 + 8 \times 3^4 + 6 \times 3^4}{24} = 333. \tag{14}$$

- Polya's theory to deal with extra requirements (how many yellow, how many red, etc.).