Review of Graph Theory

Basics of simple graphs

- Definition. A simple graph is a pair of two finite sets: A set of "vertices" $V = \{a, b, c, ...\}$, and a set of "edges" E whose elements are subsets of size 2 of the set V. We denote the graph by $G = (V, E)$.

- Order. Let $G = (V, E)$ be a simple graph. Then $n := |V|$ is called the "order" of the graph.

- Degree and degree sequence.

  - Degree of a vertex: $\deg(v)$ is the number of edges connected to v.

  - Degree sequence: List the n degrees decreasingly: $(d_1, ..., d_n), d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n \geqslant 0$.

- Facts.

  - Let G be a simple graph of order n. Then $|E| \leqslant \frac{n(n-1)}{2}$.

  - $d_1 + \cdots + d_n = 2|E|$.

  - $V + F = E + 2$ for planar graphs.

Chromatic number and chromatic polynomials

- Definitions.

  - Chromatic number. $\chi(G)$ is the smallest number of colors needed to color the vertices of G so that the two ends of any edge are colored differently.

  - Chromatic polynomial. A polynomial $P_G(k)$ such that when $k \in \mathbb{N}$, $P_G(k)$ is the number of different ways to use k colors to color G with the vertices already marked $v_1, ..., v_n$.

- Facts.

  - $1 \leqslant \chi(G) \leqslant n$.

  - $\chi(G) = 1 \Longleftrightarrow G = N_n$ is the null graph.

  - $\chi(G) = n \Longleftrightarrow G = K_n$ is the complete graph.

  - $\chi(G) \leqslant \Delta + 1$ where Δ is the maximum degree: $\Delta := \max\{d_1, ..., d_n\}$.

  - $\chi(G)$ is the smallest natural number such that $P_G(k)$ is non-zero. That is

$$\chi(G) = \min\{k | k \in \mathbb{N}, P_G(k) \neq 0\}. \tag{1}$$

- Deletion-Contraction. Let $e = \{a, b\} \in E$ for $G = (V, E)$.

$$P_G(k) = P_{G_D}(k) - P_{G_C}(k) \tag{2}$$

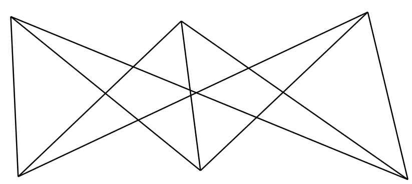
  where G_D is obtained from G by deleting e, and G_C is obtained from G by identifying (merging) a, b.

More examples and exercises

Example 1. Does there exist a graph of order 5 whose degree sequence is $(4,4,3,2,2)$?

Solution. No. As the graph has order 5, the two vertices with degrees 4,4 are connected to every other vertex. If we remove both vertices and the corresponding edges, we are left with a graph of order 3 with degree sequence $(1,0,0)$ which is not possible as $1+0+0=1$ is odd.

Example 2. Prove that the following graph G is not planar.



Proof. Assume the contrary, that is there is a way to draw G in a plane so that no edges crossing. We see that $V=6$ and $E=9$. It is easy to see that there is no triangle therefore every face has at least four vertices. Consequently we have

$$4\,F \leqslant d_1 + \cdots + d_6 = 2\,E = 18 \Longrightarrow F \leqslant 4. \tag{3}$$

But then $V+F \leqslant 10 < 11 = E+2$, contradicting Euler's formula. Therefore G is not planar. □

Example 3. Given n points on the circumference of a circle, what is the maximum number R of regions that can be formed when they are joined in pairs?

Solution. Clearly R is reached when no three of the lines joining three different pairs of points intersect at the same point.

We treat the resulting configuration as a graph[1] whose vertices are the n points and all the intersections inside the circle, edges are the line or arc segments with vertices at the ends, and the faces are the regions created, together with the region outside the circle. Thus we have $F=R+1$.

On the other hand, we see that each vertex is either one of the n points, or the intersection of two lines connecting two different pairs of the n points. Consequently we have

$$V = n + \binom{n}{4}. \tag{4}$$

To find out the number of edges, we notice that each vertex on the circle has degree $n+1$ while each interior vertex has degree 4. As $\sum \deg(v) = 2\,|E|$ holds even for general graphs, we have

$$|E| = \frac{1}{2}\left[n\,(n+1) + \binom{n}{4}4\right]. \tag{5}$$

Now Euler's formula gives

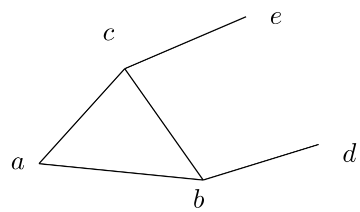
$$F = E + 2 - V = 2 + \binom{n}{2} + \binom{n}{4}. \tag{6}$$

Consequently $R = 1 + \binom{n}{2} + \binom{n}{4}$.

1. Note that the graph is not simple.

Exercise 1. Prove that $R = \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4}$. (Hint:[2])

Example 4. Calculate $P_G(k)$ for the following graph G, and determine its chromatic number $\chi(G)$.



Then calculate how many different ways are there to color the graph if we erase if the vertices are unmarked.

Solution.

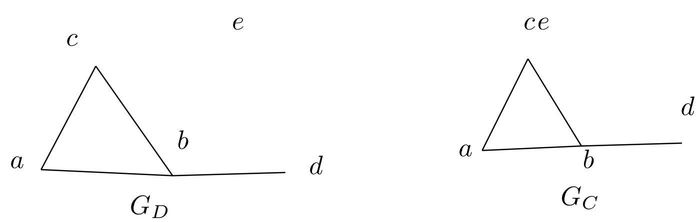
- $P_G(k)$. We calculate $P_G(k)$ in two ways.

  I. Deletion-contraction.

  - First take the edge $\{c, e\}$. We have

    $$P_G(k) = P_{G_D}(k) - P_{G_C}(k) \tag{7}$$

    where

    Exercise 2. Show that $P_{G_D}(k) = k\, P_{G_C}(k)$ without actually calculating either of the polynomials.

    We easily see that

    $$P_G(k) = (k-1)\, P_{G_C}(k). \tag{8}$$

  - Next apply deletion-contraction to $\{b, d\}$ in G_C. We see that

    $$P_{G_C}(k) = P_{G_{CD}}(k) - P_{G_{CC}}(k) = (k-1)\, P_{G_{CC}}(k). \tag{9}$$

    As G_{CC} is K_3, there holds

    $$P_{G_{CC}}(k) = k\,(k-1)\,(k-2). \tag{10}$$

  Putting things together we have

  $$P_G(k) = k\,(k-1)^3\,(k-2). \tag{11}$$

2. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

II. Direct calculation.

The graph G is actually simple enough to calculate $P_G(k)$ directly. There are $k\,(k-1)\,(k-2)$ different ways to color the triangle a, b, c. For each of the colorings, there are further $k-1$ choices for e and $k-1$ choices for d. Therefore

$$P_G(k) = k\,(k-1)^3\,(k-2). \tag{12}$$

- $\chi(G)$.

  It is clear that $P_G(1) = P_G(2) = 0$ while $P_G(3) = 24$. Therefore $\chi(G) = 3$.

- When the vertices are unmarked. We apply Burnside's Lemma.

  Exercise 3. Does Polya's theory apply here? Why?

  By the above calculation $|X| = P_G(k) = k\,(k-1)^3\,(k-2)$. The automorphism group G consists of two elements, the identity i, and the permutation $g = (a)(bc)(de)$. As there is an edge connecting b, c, there is no coloring that does not change under the action of g. Consequently $|X_g| = 0$. Therefore the answer is

$$\frac{P_G(k)}{2} = \frac{k\,(k-1)^3\,(k-2)}{2}. \tag{13}$$

Exercise 4. Prove that a graph with chromatic number equal to k has at least $\binom{k}{2}$ edges.

Exercise 5. What is the chromatic number of the graph obtained from K_n by removing one edge?

Exercise 6. What is the chromatic polynomial of the graph obtained from K_n by removing one edge? (Hint:[3])

Exercise 7. Prove that the polynomial $k^4 - 4\,k^3 + 3\,k^2$ is not the chromatic polynomial of any graph. (Hint:[4])

Exercise 8. Let G be a planar graph in which every vertex has the same degree k. Prove that $k \leqslant 5$. (Hint:[5])

Exercise 9. Let G be a planar graph of order $n \geqslant 2$. Prove that G has at least two vertices whose degrees are at most 5.

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3. Study the deletion-contraction relation.
4. Any $P_G(k)$ is an increasing function of k.
5. Euler's formula.