The 5-Color Theorem

Vertex-Edge-Face relation for planar graphs

To prove that every planar graph can be colored with at most five colors, we need to first understand what is special about planar graphs, as if we drop the "planar" assumption, then there are many graphs that cannot be colored by five or less colors, such as K_5, the complete graph of order 5.

Definition 1. (Face) *Any planar graph, not necessarily simple, divides the plane into one exterior region and many interior regions. Each such region (including the exterior one) is called a* face.

Theorem 2. (Euler) *Let G be a connected planar graph, not necessarily simple. Let V, E, F denote the number of vertices, edges, and faces of G. Then $F + V = E + 2$.*

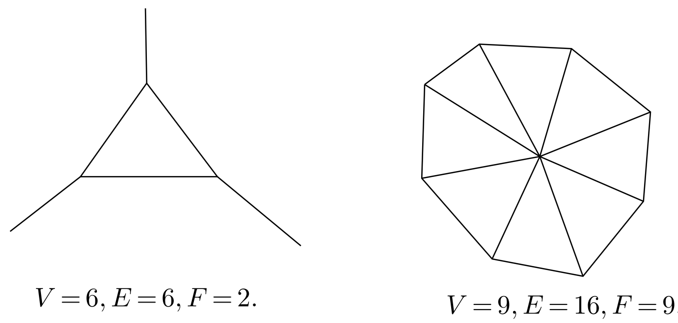

Figure 1. $F + V = E + 2$

Exercise 1. Does Euler's formula still hold if the graph is not connected? Justify your claim.

Example 3. We prove that K_5 is not planar. Assume the contrary. Then K_5 can be drawn on the plane without any edges crossing each other. Then $V = 5, E = 10$. We do not know F but it can be estimated as follows.

- On one hand every face is a (curved) polygon with at least three vertices.
- On the other hand, the number of faces adjacent to a certain vertex is the same as the degree of that vertex.

Combining the above we see that the summing up all the degrees would be counting each face at least three times, that is

$$3F \leqslant d_1 + \cdots + d_n = 2E \Longrightarrow F \leqslant 6. \tag{1}$$

But then

$$F + V \leqslant 11 < 12 = E + 2 \tag{2}$$

we have contradiction.

Example 4. Given n points on the circumference of a circle, what is the maximum number R of regions that can be formed when they are joined in pairs?

Solution. Clearly R is reached when no three of the lines joining three different pairs of points intersect at the same point.

We treat the resulting configuration as a graph[1] whose vertices are the n points and all the intersections inside the circle, edges are the line or arc segments with vertices at the ends, and the faces are the regions created, together with the region outside the circle. Thus we have $F = R + 1$.

1. Note that the graph is not simple.

On the other hand, we see that each vertex is either one of the n points, or the intersection of two lines connecting two different pairs of the n points. Consequently we have

$$V = n + \binom{n}{4}. \tag{3}$$

To find out the number of edges, we notice that each vertex on the circle has degree $n+1$ while each interior vertex has degree 4. As $\sum \deg(v) = 2\,|E|$ holds even for general graphs, we have

$$|E| = \frac{1}{2}\left[n\,(n+1) + \binom{n}{4} 4\right]. \tag{4}$$

Now Euler's formula gives

$$F = E + 2 - V = 2 + \binom{n}{2} + \binom{n}{4}. \tag{5}$$

Consequently $R = 1 + \binom{n}{2} + \binom{n}{4}$.

Exercise 2. Prove that $R = \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4}$. (Hint:[2])

The 6-color theorem

LEMMA 5. *In any planar graph there is a vertex with degree $\leqslant 5$.*

Proof. Assume the contrary, that is the degree of every vertex is $\geqslant 6$. We then have

$$3\,F \leqslant d_1 + \cdots + d_n = 2\,E \tag{6}$$

and

$$6\,V \leqslant d_1 + \cdots + d_n = 2\,E. \tag{7}$$

Consequently

$$F + V \leqslant \frac{2\,E}{3} + \frac{2\,E}{6} = E < E + 2. \tag{8}$$

Contradiction. □

THEOREM 6. *Any connected planar graph can be colored by 6 colors or less.*

Proof. We prove by induction on the order n. Clearly the conclusion holds when $n \leqslant 6$ so the base case is settled.

Now assume that any connected planar graph of order n can be colored by 6 colors or less. Let G be a planar graph with $n+1$ vertices.

By Lemma 5 there is a vertex with degree five or less. We mark this vertex v_{n+1} and the remaining vertices $v_1, \ldots, v_n$, so that v_{n+1} is directly connected to $v_1, \ldots, v_l$ for some $l \leqslant 5$.

Now let G' be the graph obtained from G by deleting the vertex v_{n+1} and the edges $\{v_{n+1}, v_1\}$, ..., $\{v_{n+1}, v_l\}$. By the induction hypothesis G' can be colored by 6 colors.

Finally, as $l \leqslant 5$, there is a color in the six colors that is different from the colors of $v_1, \ldots, v_l$. We color v_{n+1} by this color and have obtained a 6-coloring of G.

Thus ends the proof. □

The 5-color theorem

THEOREM 7. *Any connected planar graph can be colored by 5 colors or less.*

Proof. We prove the theorm by induction on the order n. The base case is trivial as when $n \leqslant 5$ clearly 5 colors suffice. Now assume that every planar graph of order n can be colored by 5 colors. Let G be a graph of order $n+1$.

In light of Lemma 5, there is a vertex in G with degree $\leqslant 5$. Similar to the proof of Theorem 6, we can show that if there the degree of this vertex is no more than 4, then we can color G by 5 colors.

2. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Exercise 3. Prove this.

In the following we assume the degree of this vertex is exactly 5. We denote this vertex by v_{n+1} and the five vertices connected to it by $v_1, ..., v_5$. We denote the five colors by r(ed), g(reen), b(lue), y(ellow), m(agenta).

Now let $G' = (V', E')$ where $V' = \{v_1, ..., v_n\}$ and $E' = E - \{\{v_1, v_{n+1}\}, \{v_2, v_{n+1}\}, ..., \{v_5, v_{n+1}\}\}$. As the order of G' is n, it can be colored by 5 colors. We take one such coloring. If in this coloring the five vertices $v_1, ..., v_5$ are colored by 4 colors or less, then we can color v_{n+1} by (one of) the unused colors. Thus in the following we assume that $v_1, ..., v_5$ are all colored differently. Without loss of generality, assume that $v_1 \leftarrow r$, $v_2 \leftarrow g$, $v_3 \leftarrow b$, $v_4 \leftarrow y$, $v_5 \leftarrow m$.

Consider v_1, v_2. Let C be all the vertices in G' that is connected to v_1 by a path involving vertices colored r, g only. There are two cases.

i. $v_2 \notin C$. We re-color G' as follows:

- For $v \in C$, if it is colored red, color it green and vice versa;
- For $v \notin C$, keep the original color.

Clearly this is still a 5-coloring of G'. As now v_1 is green, we can color v_{n+1} red and obtain a 5-coloring of G.

ii. $v_2 \in C$.

Simiar considerations for other pairs $\{v_i, v_j\}, i, j \in \{1, 2, 3, 4, 5\}, i \neq j$ reveals that in all cases except one we can always color G by 5 colors. The remaining case is:

- There is a path C_{ij}, involving vertices colored by the two colors for v_i, v_j, connecting v_i, v_j for every pair $i, j \in \{1, 2, 3, 4, 5\}, i \neq j$.

But this is not possible since K_5 is not planar.

Exercise 4. Justify this claim.

Thus we finish the proof. □