Chromatic Polynomials

Consider the following question: Given a graph G and k colors, how many different ways are there to color G with these colors?

Definition and examples

Definition 1. (Chromatic Polynomial) *Let G be a simple graph of order n with vertices marked $v_1, \ldots, v_n$. For every $k \in \mathbb{N}$, let $P_G(k)$ be the number of ways to color the graph G with k colors so that any two vertices connected by an edge are colored differently. $P_G(k)$ is called the* chromatic polynomial *of the graph G.*

Remark 2. In other words, $P_G(k)$ is the number of functions σ from $V = \{v_1, \ldots, v_n\}$ to $\{1, 2, \ldots, k\}$ so that $\sigma(v_i) \neq \sigma(v_j)$ if $\{v_i, v_j\} \in E$.

Remark 3. It is clear that $P_G(k) = 0$ when $k < \chi(G)$. Thus any root to $P_G(k)$ gives a lower bound for $\chi(G)$. The original motivation of Birkhoff to introduce the idea of chromaric polynomial in 1912 is to solve the four-color problem as follows:

i. Characterize chromatic polynomials for planar graphs;

ii. Characterize the roots of these polynomials.

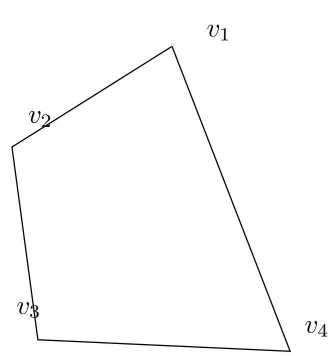
Unfortunately, to this day we are still stuck at i.

Example 4. Let N_n be the null graph of order n. We have $P_{N_n}(k) = k^n$.

Example 5. Let K_n be the complete graph of order n. We have $P_{K_n}(k) = k\,(k-1)\cdots(k-(n-1))$, as there are k colors available for v_1, then $k-1$ colors available for v_2, ... , $k-(n-1)$ colors available for v_n.

Exercise 1. Let G be any simple graph of order n. Then $P_{K_n}(k) \leqslant P_G(k) \leqslant P_{N_n}(k)$.

Example 6. [1]We try to calculate $P_G(k)$ for the graph



[1]. Note that this is a motivational example. Later we will discuss better ways of calculating $P_G(k)$ for graphs.

First we notice that $P_G(1)=0$ and $P_G(2)=2$. In general, for $k>2$, let's assume that one of the colors is red. Then $P_G(k)=k\,R(k)$ where $R(k)$ is the number of ways coloring the graph with v_1 colored red. To calculate $R(k)$, we divide into two cases (and then apply the sum rule).

- v_3 is also colored red. In this case v_2 has $k-1$ choices and v_4 too. Thus we have $(k-1)^2$.

- v_3 is not colored red. In this case v_3 has $k-1$ choices and v_2, v_4 have $(k-2)$ choices each. Consequently we have $(k-1)\,(k-2)^2$.

Thus overall we have

$$P_G(k)=k\,[(k-1)^2+(k-1)\,(k-2)^2]=k\,(k-1)\,(k^2-3\,k+3)=k^4-4\,k^3+6\,k^2-3\,k. \tag{1}$$

Remark 7. Note that we can become more confident of our calculation of $P_G(k)$ by checking $P_G(k)$ for small values of k:

$$P_G(0)=0; \qquad P_G(1)=1; \qquad P_G(2)=2. \tag{2}$$

Properties

Theorem 8. (Deletion-Contraction) *Let $G=(V,E)$ be a simple graph. Let $e=\{a,b\}\in E$. Let G_D be the graph obtained from G by deleting the edge e, and let G_C be the graph obtained from G by identifying the two vertices a,b*[2]*. Then*

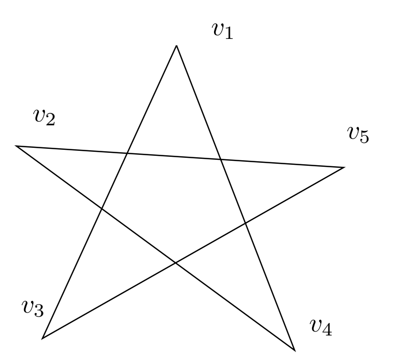
$$P_G(k)=P_{G_D}(k)-P_{G_C}(k). \tag{3}$$

Proof. It is clear that $P_G(k)=P_{G_D}(k)-N(k)$ where

$$N(k):=\text{Number of ways coloring } G \text{ with } k \text{ colors such that } a,b \text{ are colored the same.} \tag{4}$$

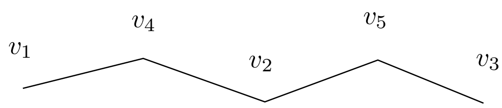
It is now clear that $N(k)=P_{G_C}(k)$. □

Example 9. We calculate $P_G(k)$ for the graph

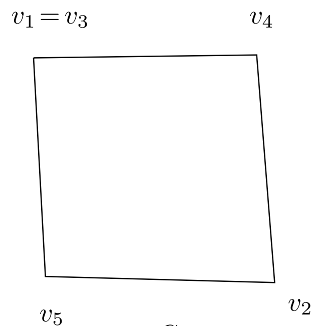


through deletion-contraction. Take the edge $\{v_1, v_3\}$. It is easy to see that G_D is a chain, and G_C can be represented as a square cycle.

2. or "merging" a,b. Thus the edge e disappears. Also if both $\{c,a\},\{c,b\}\in E$, then the two edges also "merge" into one edge in the graph G_C.



G_D



G_C

It is easy to see that

$$P_{G_D}(k) = k\,(k-1)^4. \tag{5}$$

On the other hand, we have

$$P_{G_C}(k) = k\,[(k-1)^2 + (k-1)(k-2)^2] = k\,(k-1)\,(k^2 - 3\,k + 3). \tag{6}$$

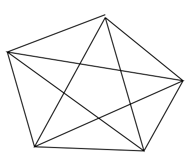
Thus we have

$$P_G(k) = P_{G_D}(k) - P_{G_C}(k) = k^5 - 5\,k^4 + 10\,k^3 - 10\,k^2 + 4\,k. \tag{7}$$

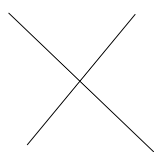
Exercise 2. Prove that the cycle with length n has $P_{C_n}(k) = (k-1)^n + (-1)^n\,(k-1)$. (Hint:[3])

Exercise 3. Let W_n be the *wheel* of $n+1$ vertices, that is a cycle of n vertices with the $n+1$-th vertex connected to them all. Find $P_{W_n}(k)$.

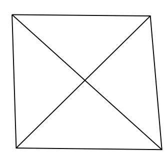
Exercise 4. Calculate $P_G(k)$ for the following graphs.



G_1



G_2



G_3

Answer: [4]

Exercise 5. Prove the following "Two-Pieces Theorem".

> **Theorem. (The Two-Pieces Theorem)** *Let the vertex set of G be partitioned into disjoint sets V_1, V_2 such that no edges in G joins a vertex in V_1 to a vertex in V_2. Let G_1, G_2 be the subgraphs generated by V_1 and V_2 respectively. Then $P_G(k) = P_{G_1}(k)\,P_{G_2}(k)$.*

Theorem 10. *$P_G(k)$ is a polynomial of k with degree n and integer coefficients.*

3. Induction.

4. G_1: $k^5 - 10\,k^4 + 35\,k^3 - 50\,k^2 + 24\,k$; G_2: $k^5 - 4\,k^4 + 6\,k^3 - 4\,k^2 + k$; G_3: $k^5 - 8\,k^4 + 24\,k^3 - 31\,k^2 + 14\,k$.

Proof. We prove through induction on the order n. When $n=1$ the claim is clearly true. Now we assume that $P_G(k)$ is a polynomial of k with degree n and integer coefficients for every simple graph of order n.

Let $G=(V,E)$ be a simple graph of order $n+1$. If $E=\varnothing$ then G is the null graph and $P_G(k)=k^{n+1}$. If $E\neq\varnothing$, let $e=\{a,b\}\in E$. By the deletion-contraction theorem we have

$$P_G(k)=P_{G_D}(k)-P_{G_C}(k). \tag{8}$$

Note that as G_C has order n, $P_{G_C}(k)$ is a polynomial of degree n. On the other hand, G_D is a simple graph of order $n+1$ but with one less edge than G.

If G_D has no edges then $P_{G_D}(k)=k^{n+1}$ and the proof ends. If not, application of the deletion-contraction theorem to G_D yields

$$P_G(k)=P_{G_{DD}}(k)-P_{G_{DC}}(k)-P_{G_C}(k) \tag{9}$$

with $P_{G_{DC}}(k)$ also a degree n polynomial.

It is clear now that we can keep doing this until there is no edge left. At the end of the day we have

$$P_G(k)=P_{G_{D\cdots D}}(k)-P_{G_{D\cdots DC}}(k)-\cdots-P_{G_{DC}}(k)-P_{G_C}(k) \tag{10}$$

with $P_{G_{D\cdots D}}(k)=k^{n+1}$ and all the other polynomials of degree n.

Thus ends the proof. □

Exercise 6. Prove that the constant term of $P_G(k)$ is 0.

Exercise 7. Prove that the lead coefficient of $P_G(k)$ is always 1.

Exercise 8. Prove that unless $P_G(k)=k^n$, the sum of the coefficients is zero.

Exercise 9. Prove that the coefficient of k^{n-1} in $P_G(k)$ is the negative of the number of edges.

Exercise 10. Show that the following cannot be chromatic polynomials.

a) k^8-1;

b) k^5-k^3+2k;

c) $2k^3-3k^2$;

d) k^3+k^2+k;

e) k^3-k^2+k;

f) $k^4-3k^3+3k^2$;

g) $k^9+k^8-k^7-k^6$.

Exercise 11. Prove that $P_G(k)\leqslant k(k-1)^{n-1}$ for any positive integer k, if G is connected with n vertices.

Exercise 12. Prove or disprove: If G' is a subgraph of G, then $P_{G'}(k)|\,P_G(k)$.

Exercise 13. Prove that if K_r is a subgraph of G, then $k(k-1)\cdots(k-r+1)|\,P_G(k)$. (Hint:[5])

5. Note that $P_G(i)=0$ for $i=0,1,2,\ldots,r-1$.