Chromatic and Clique Numbers

Definition 1. (Chromatic Number) *Let G be a graph. The chromatic number $\chi(G)$ is the smallest number of colors needed to color the vertices such that any two ends of an edge are colored differently.*

Theorem 2. *Let G be a graph of order $n \geqslant 1$. Then*

$$1 \leqslant \chi(G) \leqslant n. \tag{1}$$

Furthermore, $\chi(G) = n$ if and only if $G = K_n$.

Proof. (1) is trivial. It is also clear that $\chi(K_n) = n$. In the following we prove that $\chi(G) = n \Longrightarrow G = K_n$. Assume otherwise. Then there is one vertex with degree at most $n - 2$. We denote this vertex by v_n and the other vertices $v_1, ..., v_{n-1}$, so that there is no edge between v_n and v_{n-1}. We color $v_1, ..., v_{n-1}$ with $n - 1$ colors $C_1, ..., C_{n-1}$, and v_n with C_{n-1}.

Now let $e = \{v_i, v_j\} \in E$ be arbitrary. If $i, j \in \{1, ..., n - 1\}$, then clearly the two vertices are colored differently. On the other hand, if one of them is v_n, then the other cannot be v_{n-1} and they are still colored differently.

Thus we have colored G with $n - 1$ colors. Contradiction. □

Exercise 1. Prove that $\chi(G) = 1$ if and only if G is the null graph, that is a graph with no edges.

Definition 3. (Clique Number) *Let G be a graph. Its clique number $C(G)$ is defined to be the largest $p \in \mathbb{N}$ such that the order p complete graph K_p is a subgraph of G.*

Exercise 2. Prove that $\chi(G) \geqslant C(G)$.

Example 4. It is possible for $\chi(G) > C(G)$. Let $G = (V, E)$ where $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{e, a\}\}$. Then we have $C(G) = 2$ but $\chi(G) = 3$.

Theorem 5. *Let $G = (V, E)$ be a graph. Then*

$$\chi(G) \leqslant \Delta + 1 \tag{2}$$

where $\Delta := \max_{v \in V} \deg(v)$ is the maximum degree of the vertices of G.

Proof. Let $|V| = n$. We prove the claim through induction on n. It is clear that the claim holds when $n = 1$, 2 or even 3. Thus in the following we prove $n \Longrightarrow n + 1$.

Assume that the claim is true for any graph with n vertices. Let $G = (V, E)$ with $V = \{v_1, ..., v_{n+1}\}$. Let G' be the graph obtained from G through erasing v_{n+1} and all edges connected to it. Then G' is of order n, and furthermore $\Delta' \leqslant \Delta$. Consequently G' can be colored by at most $\Delta + 1$ colors.

Now we add v_{n+1} and the corresponding edges back. As $\deg(v_{n+1}) \leqslant \Delta$, there are at most Δ vertices from $\{v_1, ..., v_n\}$ connected to v_{n+1}. Consequently there is a color among the $\Delta + 1$ colors that is not used on the vertices connected to v_{n+1}, and thus can be used to color v_{n+1}. □

Example 6. Clearly $\chi(G) = \Delta + 1$ for $G = K_n$.

Exercise 3. Show that there is $G \neq K_n$ for any n, with $\chi(G) = \Delta + 1$.

Theorem 7. (Brooks) [1]*Assume that G is connected but is neither a complete graph nor a cycle graph of odd order, then $\chi(G) \leqslant \Delta$.*

Exercise 4. Show that the theorem does not hold if we drop the assumption of connectedness.

Proof. [2]First we prove the claim when $\Delta = 1$ or 2.

When $\Delta = 1$, the only possibility of G is $G = K_2$.

When $\Delta = 2$, we color G as follows. Start from a vertex a with degree 2. We color it red and color the two neighbours b, c green. Now if $\deg(b) = \deg(c) = 1$, we have G to be exactly this order three graph. If one of the degrees is 2, we can color the neighbours red and repeat the above. This procedure will halt if

- we have colored all the vertices with two colors; or
- The new neighbour is a vertex that has already been colored. In this case either this new neighbour has degree three, or G becomes a cycle.

In the following we prove the theorem when $\Delta = 3$ and leave the general case as an exercise. We prove by induction on the order n of G. When $n = 4$ clearly three colors suffice. Now assume that any graph with order n and maximum degree 3 can be colored with three colors. Consider a graph G of order $n + 1$.

Let x be a vertex in G such that $\deg(x)$ is (one of) the smallest. Let G' be the graph obtained by deleting the vertex x and all the edges containing x. By the induction hypothesis G' can be colored with three colors. If $\deg(x) \leqslant 2$, then clearly we can color x using one of the three colors. Thus in the following we assume $\deg(x) = 3$.

Let the vertices connected to x be x_1, x_2, x_3. If in the coloring of G' they are already colored by two or less colors then we can color x by the third color. If not, we will show in the following that we can always re-color G' so that $x_1, \ldots, x_3$ are colored by at most two colors.

Assume that x_1 is colored red, x_2 green, x_3 yellow. We also denote by $G'_{r,g}$ the graph consisting of all the vertices in G' colored either red or green, and all the edges between them. Denote $G'_{g,y}, G'_{y,r}$ similarly.

First we claim that there is a path in $G'_{r,g}$ connecting x_1 and x_2. Assume otherwise. Let V_1 be all the vertices in $G'_{r,g}$ that is connected to x_1 by edges in $G'_{r,g}$, then none of the vertices in V_1 is connected to x_2 in $G'_{r,g}$. In other words, if a vertex in V_1 is connected to x_2 in G', then along the path there is a vertex colored yellow. Now we see that we can re-color G' as follows:

i. For vertices in V_1, color every red vertex green and every green vertex red;

ii. For other vertices in G, do not change the coloring.

Then we have a three-coloring of G' with x_1, x_2 both green and x_3 yellow, and can color x red to obtain a three-coloring of G.

Now assume that there is a path $L_{r,g}$ connecting x_1 and x_2 in $G'_{r,g}$. Similarly there is a path $L_{g,y}$ in $G'_{g,y}$ connecting x_2, x_3, and there is a path $L_{y,r}$ in $G'_{y,r}$ connecting x_3, x_1. We notice that

1. R. L. Brooks: *On coloring the nodes of a network*, Proc. Cambridge Philos. Soc., 37 (1941), 194–197.

2. The proof is adapted from L. S. Melnikov and V. G. Vizing, *New proof of Brooks' theorem*, Journal of Combinatorial Theory 7, 289–290, 1969.

These paths do not "cross" one another at other vertices than x_1, x_2, x_3. To see this, assume that there is a vertex y different from x_1, x_2, x_3 belonging to $L_{r,g}$ and $L_{g,y}$. Thus y must be colored green. Then its two neighbours in $L_{r,g}$ must be colored red and its two neighbours in $L_{g,y}$ must be colored yellow. But this means $\deg(y) \geqslant 4$. Contradiction.

Now consider an arbitrary vertex y on $L_{r,g}$ that is different from x_1, x_2 (if there is any such vertex). Assume that it is colored green. Then both its neighbours in $L_{r,g}$ are colored red. As the degree at this vertex is 3, it is connected to one more vertex in G'. If this vertex belongs to $G'_{r,g}$ then it must also be colored red. Now we re-color y be yellow. This breaks the path $L_{r,g}$ and consequently the resulting graph can be further re-colored so that x_1, x_2 are of the same color.

Now we are left with the following case: Every vertex on $L_{r,g}$ that is different from x_1, x_2 is connected to a vertex that is colored yellow, and similar claims hold for $L_{g,y}$ and $L_{y,r}$. This implies, noticing that there are already three edges from x_1, x_2, x_3, that $L_{r,g}$ is the only path connecting x_1, x_2 in $G_{r,g'}$, and similar claims hold for $L_{g,y}$ and $L_{y,r}$.

Now the situation we need to deal with is as follows. There is a unique path $L_{r,g}$ connecting x_1, x_2 with the vertices along the path colored red and green alternatively, a unique path $L_{g,y}$ connecting x_2, x_3 in with the vertices colored green and yellow alternatively, and a unique path $L_{y,r}$ connecting x_3, x_1 with the vertices along it colored yellow and red alternatively. Since G does not contain K_4, at least one of these paths has length >1. Assume that it is $L_{r,g}$. Then there is a vertex $y_1 \in L_{r,g}$ such that it is connect to x_1 and $y_2 \in L_{r,g}$ connected to x_2. Clearly y_1 is green and y_2 is red.

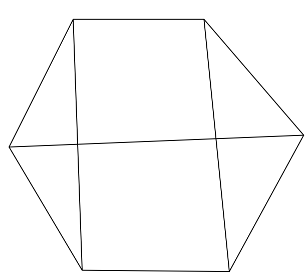
We re-color G' as follows.

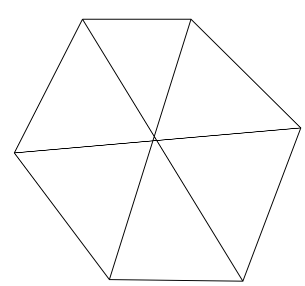
i. Along $L_{y,r}$, color red vertices yellow and yellow vertices red.

ii. Leave the colors of other vertices unchanged.

This is also a legitimate three-coloring of G', and x_1 is now yellow. Now let L be an arbitrary path in G' from x_3 to x_2. We see that there is at least one vertex on L that is colored yellow. In other words, the path $L_{r,g}$ does not exist anymore, and we can further re-color G' to make x_3 green. Now x can be colored yellow. □

Exercise 5. Color the following two graphs with three colors.





Exercise 6. Prove Theorem 7 for the general case $\Delta \geqslant 3$.

Remark 8. Mycielski showed that there are triangle-free graphs with an arbitrarily high chromatic number. For example, the Grötzsch graph (see below) has $\chi(G) = 4$.

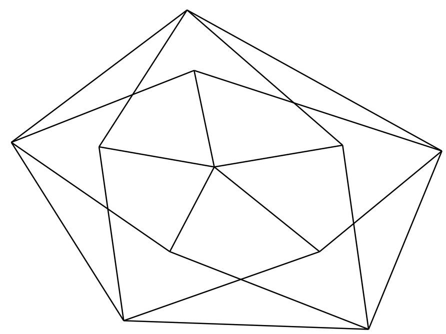


Figure 1. the Grötzsch graph