Counting with Symmetry

- Counting using Burnside's Lemma;
  1. Identify the set X;
  2. Identify the group G that acts on X;
  3. For every g, count how many $x \in X$ satisfies $g x = x$, that is calculate $|X_g|$ where $X_g := \{x \in X | g x = x\}$;
  4. The answer, the number of orbits, is given by

$$\frac{1}{|G|} \sum_{g \in G} |X_g|. \tag{1}$$

- Counting using Polya's theory. Polya's theory applies to the problem of counting how many ways there are to color a device with symmetry using m colors.
  1. Identify the group G of allowed operations;
  2. Let n be the number of objects (in one device) that are getting colored. Then every $g \in G$ is a permutation on $\{1, 2, ..., n\}$. Write every $g \in G$ in "cyclic form", count the number of cycles $c(g)$.
  3. The answer is given by

$$\frac{1}{|G|} \sum_{g \in G} m^{c(g)}. \tag{2}$$

- Examples.

Example 1. Let p be an odd prime. How many ways are there to color the edges of a regular p-gon with m colors if both rotation and flipping are allowed?

Solution. We use Polya's theory. The symmetry group for a regular p-gon with p an odd prime consists of

- The identity $i = (1)(2)\cdots(p)$. Thus $c(i) = p$.
- The $p - 1$ rotations of $\frac{2\pi}{p}, \frac{4\pi}{p}, ..., \frac{2(p-1)\pi}{p}$. As p is prime, each such rotation consists of only one cycle of length p. Thus $c(g) = 1$ for each of them.
- The p flippings around the line passing a vertex and the middle of the opposite edge. The cyclic form is $(1)(2\,(p-1))(3\,(p-2))\cdots$ so $c(g) = \frac{p+1}{2}$ for them.

So the answer is

$$\frac{m^p + p\, m^{(p+1)/2} + (p-1)\, m}{2\, p}. \tag{3}$$

Exercise 1. Let p, q be odd primes. Let $n = p\, q$. How many ways are there to color the edges of a regular n-gon with m colors if only rotation is allowed?

Exercise 2. (If you know elementary number theory) Prove that $2\, p | m^p + p\, m^{(p+1)/2} + (p-1)\, m$ for all $m \in \mathbb{N}$.

Example 2. A decimal sequence is a sequence whose digits are 0,1,2,...,9. The digits 0, 1, 6, 8, 9 become 0, 1, 9, 8, 6 respectively, when they are turned upside down. We see that two decimal sequences are equivalent if one can be transformed into the other by a 180 degree rotation. Find the number of different n-digit decimal sequences.

Wrong solution. Let X be all n-digit decimal sequences, there are 10^n such sequences. The group G of allowed operations has two elements, identity i, and the 180 degree rotation r.

- Obviously $X_i = X$ so $|X_i| = 10^n$;
- X_r. It is clear that any decimal sequence in X_r only consists of $0, 1, 9, 8, 6$.
  - $n = 2\, k + 1$, odd. In this case the middle digit can be $0, 1, 8$ and the last k digits is the 180 degree rotation of the first k digits. Thus there are 3×5^k such sequences.

- $n=2k$, even. In this case the last k digits is the 180 degree rotation of the first k digits. Thus there are 5^k such sequences.

Thus

$$\text{Ans} = \begin{cases} \frac{1}{2}(10^n + 3\times 5^k) & n=2k+1 \\ \frac{1}{2}(10^n + 5^k) & n=2k \end{cases}. \tag{4}$$

We note that clearly the answer is not an integer in either case!!

Solution. The mistake we made in the above "wrong solution" is that G does not "act" on X. If a sequence involves 2,3,4,5, or 7, then rotating it does not yield another sequence in X. To fix this, we define X to be n-digit sequences consisting of 0,1,9,8,6 only. Application of Burnside's Lemma now gives the number of n-digit sequences consisting of 0,1,9,8,6 only:

$$\begin{matrix} \frac{1}{2}(5^n + 3\times 5^k) & n=2k+1 \\ \frac{1}{2}(5^n + 5^k) & n=2k \end{matrix}. \tag{5}$$

The final answer is then

$$10^n - 5^n + \begin{cases} \frac{1}{2}(5^n + 3\times 5^k) & n=2k+1 \\ \frac{1}{2}(5^n + 5^k) & n=2k \end{cases}. \tag{6}$$

Exercise 3. Find the number of different n-digit numbers under the same symmetry.

Example 3. [1]How many different ways are there to color the vertices of a pyramid (that is free to move in space) with white and blue such that three vertices are white and two are blue? Use Polya's theory to solve this.

Solution. First we identify the symmetry group of a pyramid. We notice that among the 5 vertices, the "top" vertex is special. Therefore the symmetry group of a pyramid is a subgroup of the symmetry group of the base square. Furthermore we notice that "flipping" is not allowed here. Thus

$$G = \{i, r_1, r_2, r_3\} \tag{7}$$

where r_1, r_2, r_3 are counter-clockwise rotations of 90, 180, 270 degrees around the line passing the "top" vertex and the center of the "base" square. Now mark the top vertex 1 and the base vertices $2,3,4,5$ counterclockwise.

We have

- $i=(1)(2)(3)(4)(5)$ yields $(w+b)^5$;
- $r_1=(1)(2345)$ yields $(w+b)(w^4+b^4)$;
- $r_2=(1)(24)(35)$ yields $(w+b)(w^2+b^2)^2$;
- $r_3=(1)(2543)$ yields $(w+b)(w^4+b^4)$.

Therefore the answer is the coefficient of $w^3 b^2$ in the expansion of

$$\frac{(w+b)^5 + 2(w+b)(w^4+b^4) + (w+b)(w^2+b^2)^2}{4} = 3. \tag{8}$$

Example 4. [2]Show that if n is a positive integer, then 24 divides $n^8 + 17n^4 + 6n^2$.

1. http://www.math.ualberta.ca/~isaac/math421/w06/fn_review.pdf.
2. http://www.math.ualberta.ca/~isaac/math421/w06/fn_review.pdf.

Proof. Let's color the vertices of a cube by n colors. Recall the symmetry group of a cube (as a permutation group of the eight vertices):

- $i = (1)(2)(3)(4)(5)(6)(7)(8)$ yields n^8;
- Rotations of 90 and 270 degrees around the line passing centers of faces (there are 6 of them): $(1234)(5678)$ and similar ones. These yield $6\,n^2$;
- Rotations of 180 degrees around the line passing centers of faces (there are 3): $(13)(24)(57)(68)$ and similar ones. These yield $3\,n^4$;
- Rotations around long diagonal of 120 and 240 degrees (there are 8): $(1)(245)(7)(368)$ and similar ones. These yield $8\,n^4$;
- Rotations of 180 degrees around the lines passing middle points of edges (there are 6): $(12)(35)(46)(78)$ and similar ones. These yield $6\,n^4$.

Thus there are

$$\frac{n^8 + 17\,n^4 + 6\,n^2}{24} \tag{9}$$

different ways to color the vertices of the cube. As this number must be an integer, 24 must divide $n^8 + 17\,n^4 + 6\,n^2$. □