Polya Counting

Recall our (upgraded) counting procedure for the coloring a device of n balls connected into some geometric shape through rods with m colors.

1. Get $(m^n) \cdot n$ identical balls. Every time take n of them and mark $1, 2, \ldots, n$. Color each such group of n balls differently and assemble them into the desired geometric shape. We obtain m^n devices that are colored and marked. We call this collection X.

2. Each allowed operation turns one device into another if we ignore the colors but keep the marks. These operations form a group G.

3. The number of different devices we would have after erasing the marks is given through Burnside's Lemma + Polya counting:

$$\text{Ans} = \frac{1}{|G|} \sum_{g \in G} m^{c(g)} \tag{1}$$

   where $c(g)$ is the number of cycles when g is represented in the cyclic form.

Example 1. How many ways are there to color the vertices of an equilateral triangle with three colors?

Solution. We see that $G = S_3$.

- Identity: $(1)(2)(3) \Longrightarrow c(g) = 3$;
- Rotations: $(123), (132) \Longrightarrow c(g) = 1$;
- Flippings: $(1)(23), (2)(13), (3)(12) \Longrightarrow c(g) = 2$.

Therefore

$$\text{Ans} = \frac{3^3 + 2 \times 3^1 + 3 \times 3^2}{6} = 10. \tag{2}$$

Exercise 1. How many ways are there to color 6 vertices of a regular hexagon with 4 colors? (Ans:[1])

Exercise 2. How many ways are there to color the vertices of a regular n-gon with m colors if

a) only rotation is allowed;

b) both rotation and flipping are allowed.

Example 2. How many ways are there to color the vertices of a regular tetrahedron with four colors?

1. 430.

Soluiton. We recall that the symmetry group of a regular tetrahedron consists of the identity, 4 rotations of $2\pi/3$, 4 rotations of $4\pi/3$, and 3 rotations of π. We have

- identity: $(1)(2)(3)(4) \Longrightarrow c(g) = 4$;
- rotations of $2\pi/3$: $(1)(234), (2)(341), (3)(412), (4)(123) \Longrightarrow c(g) = 2$;
- rotations of $4\pi/3$: $(1)(243), (2)(431), (3)(142), (4)(213) \Longrightarrow c(g) = 2$;
- rotations of π: $(12)(34), (13)(24), (14)(23) \Longrightarrow c(g) = 2$.

Therefore

$$\text{Ans} = \frac{4^4 + 11 \times 4^2}{12} = 36. \tag{3}$$

Example 3. Consider a cube with a bead at each vertex. We would like to use two colors R and G to color both the faces and the beads at the vertex. How many different ways are there?

Solution. Of course we could mark the six faces and eight vertices 1,2,..,14 and study every rotation of the cube as an element of the group S_{14}. However things could be made more efficient. We notice that two colorings are different if and only if either the vertices are colored differently, or the faces are colored differently, or both. Therefore the product rule applies when we consider the fixed elements in application of Burnsides' Lemma. More precisely, the set of all possible colorings (when the vertices and the faces are all marked) is $X = X^f \times X^v$ where X^f is the collection of possible colorings of the faces and X^v all possible colorings of the vertices. We have, for every $g \in G$ the symmetry group of the cube,

$$X_g = X_g^f \times X_g^v \Longrightarrow |X_g| = \left|X_g^f\right| |X_g^v|. \tag{4}$$

Thus we have (Here we use the shorthand: $(1)^2(4)^1$ means there are two 1-cycles and one 4-cycle.

$g \in G$	Faces	Vertices
i	$(1)^6$	$(1)^8$
3 rotation by $\pi/2$ around the line connecting face centers	$(1)^2(4)^1$	$(4)^2$
3 rotation by π around the line connecting face centers	$(1)^2(2)^2$	$(2)^4$
3 rotation by $3\pi/2$ around the line connecting face centers	$(1)^2(4)^1$	$(4)^2$
6 rotations by π around the line connecting middle points of oppositing edges	$(2)^3$	$(2)^4$
4 rotations by $2\pi/3$ around long diagonals	$(3)^2$	$(1)^2(3)^2$
4 rotations by $4\pi/3$ around long diagonals	$(3)^2$	$(1)^2(3)^2$

Finally we see that the answer is given by

$$\frac{2^{6+8} + 3 \times 2^{2+1+2} + 3 \times 2^{2+2+4} + 3 \times 2^{2+1+2} + 6 \times 2^{3+4} + 4 \times 2^{2+2+2} + 4 \times 2^{2+2+2}}{24} = 776. \tag{5}$$

Exercise 3. How many ways are there to color a regular tetrahedron using four colors for vertices, three colors for faces, and two colors for edges? (Ans:[2])

2. $(8 \times 4^2\, 3^2\, 2^2 + 3 \times 4^2\, 3^2\, 2^4 + 4^4\, 3^4\, 2^6)/12$.