Counting using Burnside’s Lemma

**Example 1.** Let’s recall the problem of coloring the 10-cart Merry-Go-Rounds with 2 colors.

Here $X$ is the set of $2^{10}$ differently colored 10-cart trains. And $G = \{g_0, \ldots, g_9\}$ where $g_k$ is the action: Cut the 10-cart train between the $k$th and the $(k+1)$th carts, and then re-connect the front half to the end of the back half.

By Burnside’s Lemma, we have

$$\text{Ans} = \frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{10} \sum_{i=0}^{9} |X_{g_i}|. \quad (1)$$

Now we calculate

- $X_0$. Since $g_0$ is the identity element, we have $X_0 = X$ and therefore $|X_0| = 1024$.
- $X_1$. A 10-cart train remains the same after the action of $g_1$ if and only if it is colored by a single color and therefore $|X_1| = 2$.
- $X_2$. A 10-cart train remains the same after the action of $g_2$ if and only if carts 1,3,5,7,9 are colored the same and 2,4,6,8,10 are colored the same. So $|X_2| = 4$.
- $X_3$. A 10-cart train remains the same after the action of $g_3$ if and only if all carts are colored the same. So $|X_3| = 2$.
- $|X_4| = 4$.
- $|X_5| = 32$.
- $|X_6| = 4$.
- $|X_7| = 2$.
- $|X_8| = 4$.
- $|X_9| = 2$.

Therefore the answer is

$$\frac{1024 + 2 + 4 + 2 + 4 + 32 + 4 + 2 + 4 + 2}{10} = 108. \quad (2)$$

**Exercise 1.** How many ways are there to color a 6-cart Merry-Go-Round with 4 colors?

**Example 2.** Let $m \in \mathbb{N}$. How many ways are there to color the 4 faces of a regular tetrahedron with $m$ colors?

**Solution.** If we mark the faces, then we can make $m^4$ differently colored tetrahedra. These would form our set $X$. On the other hand, the group $G$ that acts on $X$ would be the symmetry group of a regular tetrahedron, that is, spatial rigid movements (translation, rotation, reflection and their combinations) that leave the tetrahedron unmoved.

Now we determine the symmetry group of a regular tetrahedron. Let’s put the tetrahedron on the $x$-$y$ plane. Mark the faces 1,2,3,4 and let face 4 be the down face, and let 1,2,3 be counterclockwise when looking from above.
Then there are three transformations that leaves the tetrahedron unmoved and with face 4 still the “down” face: counterclockwise rotation around the $z$-axis by $0, 2\pi/3, 4\pi/3$. We denote them $r_{0}, r_{4,123}, r_{4,132}$. Note that $r_{0} = i$ the identity. Also the “123” in the subscript means face 1 is rotated to face 2, face 2 to face 3 (and of course face 3 to face 1). Now as the four faces are identical, we see that there are three transformations that leaves both the tetrahedron and face 3 unmoved, denote them $r_{30} = i, r_{3,124}, r_{3,142}$. Similarly we have transformations that leave faces 1 or 2 unmoved. So far we have 9 transformations:

$$i, r_{4,123}, r_{4,132}, r_{3,124}, r_{3,142}, r_{2,134}, r_{2,143}, r_{1,243}, r_{1,234}.$$

Now consider transformations that move all four faces. It is clear that we can rotate around the line passing $A$, $B$ to achieve $1 \leftrightarrow 3, 2 \leftrightarrow 4$. We denote it by $r_{13,24}$. Similarly there are $r_{14,23}, r_{12,34}$. We now have 12 transformations.

**Exercise 2.** What is $r_{4,123}r_{12,34}$?

Are there more? Assume there is. Without loss of generality, assume face 1 becomes face 2. If $2 \rightarrow 1$ then necessarily $3 \leftrightarrow 4$ and we end up with $r_{12,34}$. Thus either $2 \rightarrow 3$ or $2 \rightarrow 4$. The two cases are clearly equivalent due to the symmetry of the regular tetrahedron. Without loss of generality assume $2 \rightarrow 3$. If $3 \rightarrow 1$ then we have $r_{4,123}$. Thus the only possibility for a new transformation is $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$. We have Thus we have $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 
\rightarrow 1$ for the faces. We now show that this transformation is impossible. To see this, note that when we are at the vertex facing 4, the faces 1, 2, 3 and counter-clockwise. This orientation should not change if we stay at this vertex and move with the tetrahedron during the transformation. However, after the transformation $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$, we are at the vertex facing 1, but the faces 2, 3, 4 are not clockwise. Contradiction.

Therefore our symmetry group $G$ has 12 elements: $i, r_{4,123}, r_{4,132}, r_{3,124}, r_{3,142}, r_{2,134}, r_{2,143}, r_{1,243}, r_{1,234}, r_{12,34}, r_{13,24}, r_{14,23}$.

We now calculate $X_{g}$ for every one of these element.

- $i$. $|X_{g}| = |X| = m^{4}$.

- $r_{4,123}$. Necessarily faces 1, 2, 3 have the same color. There are two cases.
  - All 4 faces are colored the same. There are $m$ such colorings.
  - Two colors are used. There are $2 \times \binom{m}{2}$ such colorings.\(^1\)

  So $|X_{r_{4,123}}| = m(m - 1) + m = m^{2}$.

- $r_{4,132}, \ldots, r_{1,234}$. Same as the $r_{4,123}$ case.

\(^1\) After two colors are chosen, we can further choose which one to use for face 4.
• $r_{12,34}$. Faces 1,2 have the same color and faces 3,4 have the same color. There are two cases.
  ○ All 4 faces are colored the same. There are $m$ such colorings.
  ○ Two colors are used. There are $2 \times \binom{m}{2}$ such colorings.

So $|X_{r_{12,34}}| = m^2$.

• $r_{13,24}, r_{14,23}$. Same as $r_{12,34}$.

Thus we have the total number of colorings to be

$$C(m) := \frac{m^4 + 11m^2}{12}.$$  \hspace{1cm} (4)

**Exercise 3.** Prove directly (like in number theory) that $12 \mid (m^4 + 11m^2)$. (Hint: \footnote{Write as $(m^2 + 11) m^2$ and show that it is divisible by 3 through discussing $m = 3k, 3k + 1, 3k + 2$. Similarly show that it is divisible by 4 through discussing $m = 4k, 4k + 1, 4k + 2, 4k + 3$.})

**Remark 3.** We see that

$$C(2) = 5 = \binom{4 + 2 - 1}{2 - 1}, \quad C(3) = 15 = \binom{4 + 3 - 1}{3 - 1}, \quad C(4) = 36 \neq 35 = \binom{4 + 4 - 1}{4 - 1}.$$  \hspace{1cm} (5)

So it is a coincidence that in the midterm, the number of ways to color the 4 faces with 3 colors is the same as the number of ways to color 4 identical balls in 3 colors.

**Example 4.** How many ways are there to color a 4-bead bracelet with 3 colors if

a) It is not allowed to “flip” the bracelet.

b) It is allowed to “flip” the bracelet.

**Solution.** The set $X$ has $3^4$ elements.

a) When it is not allowed to “flip”, the only possible transformations are the four rotations by $0, \pi/2, \pi, 3\pi/2$. We see that the answer is

$$\frac{3^4 + 3^3 + 3^2 + 3^1}{4} = 24.$$  \hspace{1cm} (6)

b) When “flipping” is allowed, the situation is more complicated. Besides the four rotations we have also four “flippings”:
We discuss the following cases.

- **1 → 1.** As the distance (along the circle) between the beads must be unchanged, either 2 → 2, 4 → 4 which leads to the identity transformation, or 2 ↔ 4 which is “flipping” around the 1-3 axis.

- **1 → 3.** Either 4 ↔ 2 which is rotation by \(\pi\), or 4, 2 stay unmoved which is flipping around the 2-4 axis.

- **1 → 2.** Either 4 → 1 which means 2 → 3 and 3 → 4, which is rotation by \(\pi/2\); or 4 → 3 which dictates 2 → 1 and 3 → 4. This is flipping around the diagonal axis connecting the middle of 1, 2 and 3, 4.

Thus we see that these eight transformations form the symmetry group of the bracelet. It is easy to calculate the \(|X_g|\)'s and the answer is

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\frac{3^4 + 3^3 + 3^2 + 3^2 + 3^1 + 3^1}{8} = 21. \tag{7}
\]