Groups and Group Actions

We consider the following problem: How many ways are there to color n identical balls with m colors, if the n balls are embedded in a geometric structure. This geometric structure allows a certain number of actions, so that seemingly different colorings can become identical after applying one of these actions.

The way we solve such problems is as follows.

1. Turn this problem into the following equivalently one.

   Let there be an infinite supply of the n-ball device with the specified geometric structure. We color them using m colors. How many different colored device can we produce?

2. We color each such device as follows. Take n identical balls and mark them 1, 2, ..., n. Then color them with the m colors and put them in a bag. Then we take another n beads, mark and color them, and put them in another bag, and so on. As there are m^n different ways to color n marked balls, we obtain m^n bags of colored balls.

3. We assemble the balls in each bag into the required geometric structure. We obtain m^n colored devices.

4. We erase the marks on the balls.

   During this step many of the m^n colored devices become identical. We put the m^n colored devices into boxes so that devices in the same box are identical, but devices in different boxes are different.

5. The answer now is given by the number of boxes.

Four of the above five steps are trivial. The key to solving the problem is to carry out Step 4 efficiently. We clarify the situation. At the beginning of Step 4, we have a collection of objects. There is also a prescribed collection of actions. If performing one of the actions to one device produces another device, the two devices should be put into the same box.

(Maybe) Surprisingly, that this step can be carried out at all puts severe restrictions on the collection of actions. It is these restrictions that allows efficient counting for our problem.

- **Restrictions on the collection of actions**

  i. Any action in the collection can be "un-did". Imagine we have two devices x_1 and x_2. Assume that there is an action a such that when acting on x_1, it produces x_2. Thus x_1, x_2 should be in the same box from the point of view of x_1. However if there is no action in the collection that can "un-do" action a, there may not be an action that can turn x_2 into x_1. So from the point of view of x_2, it should not be in the same box as x_1. We have a problem.

     Thus for every action a, there must be an "anti-action" b that can un-do a: Application of a and then b to a device x leaves x unchanged.

  ii. For any two actions, there is a third action that is equivalent to performing the two actions in a row. Let a, b be two actions and x be a device. If we apply a to x we obtain a device y. Thus x, y should be in the same box. Now we apply b to y to obtain another device z. Thus y, z should be in the same box. But this means x, z must be in the same box, and there must be another action c that would turn x into z directly.

  Exercise 1. Prove that as a consequence of i and ii, the "do-nothing" action, which when acting on a device x always leaves x unchanged, must be in the collection.

These restrictions turn the collection of actions into a mathematical object called a "group". A group enjoys many nice properties, among which is a theorem called the Burnside's Lemma. This theorem makes it possible to count the number of colorings of devices efficiently.

Groups

Definition 1. (Group) *A group G is a set (again denoted by G) together with a binary operation which take as input two elements from G and output another element of G, such that the following holds. Note that the binary operation is usually denoted as if it is multiplication.*

i. There is an identity element i. $i\,g = g\,i = g$ for every $g \in G$.

ii. The binary operation is associative. $(g_1\, g_2)\, g_3 = g_1\, (g_2\, g_3)$*. This means we can simply write* $g_1\, g_2\, g_3$*.*

iii. For every $g \in G$*, there is another* $h \in G$ *such that* $g\, h = h\, g = i$*. This* h *is called the "inverse" of* g *and often denoted* g^{-1}*.*

Example 2. Recall that when studying the coloring of 5 identical balls connected by 4 identical rods into a straight line segment, the allowed "actions" are:

- g_0: Do nothing.
- g_1: Rotate the device 180 degrees.

We easily check that $G = \{g_0, g_1\}$ together with the following binary operation

- $g\, h$ is the applications of h first and then g,

form a group. The identity element is g_0. The inverses are $g_0^{-1} = g_0$, $g_1^{-1} = g_1$.

Exercise 2. Let $G = \{-1, 1\}$ with multiplication as the binary operation. Is it a group?

Example 3. Recall that there are 10 allowed actions for the 10-cart Merry-Go-Rounds: Turning clockwise by $k\, \theta_0$ where $\theta_0 = 2\, \pi / 10$ for $k = 0, 1, 2, ..., 9$. We denote these 10 actions by $g_0, ..., g_9$, and take the binary operation to be:

- $g_k\, g_l =$ first turn clockwise by $l\, \theta_0$ and then by $k\, \theta_0$.

It can easily be checked that $G = \{g_0, ..., g_9\}$ with this binary operation form a group.

Exercise 3. Check this.

Example 4. Let G be the set of all 2×2 matrices whose determinant is nonzero. Let matrix multiplication be the binary operation. Then G becomes a group, usually denoted $\mathrm{GL}(2, \mathbb{R})$.[1] We note that this group is not commutative, that is $g_1\, g_2 = g_2\, g_1$ may not hold.

Example 5. Let G be the actions we can apply to the following device and leave it looking the same: Three identical balls connected by three identical rods, forming an equilateral triangle. Let the binary operation be composition of actions, that is $g_1\, g_2$ means apply g_2 first and then apply g_1.

For $k = 0, 1, 2$, let $g_k :=$ turning clockwise by $2\, k\, \pi / 3$. Let $f :=$ flip the device left$\longleftrightarrow$right.

Exercise 4. Prove that $G = \{g_0, g_1, g_2, f, g_1\, f, g_2\, f\}$. (Hint:[2])

Exercise 5. Find out $f\, g_1$, and check whether G is commutative.

Group action on sets

Definition 6. (Group action) *Let* G *be a group. Let* X *be a set. An action of* G *on* X *is a rule assigning some* $y \in X$ *to every pair* $g \in G$ *and* $x \in X$*. This operation is usually denoted as if it is a multiplication, that is the result of applying* g *to* x *is denoted* $g\, x$*. The rule must satisfy the following.*

$$(g_1\, g_2)\, x = g_1\, (g_2\, x). \tag{1}$$

Exercise 6. Make sense of (1).

Remark 7. We see that our coloring problem is exactly a group action situation.

Definition 8. (Orbits) *Consider a group* G *acting on a set* X*. Let* $x \in X$*. The orbit containing* x*, denoted* $\mathrm{Orb}(x)$*, is defined as*

$$\mathrm{Orb}(x) := \{g\, x |\, g \in G\}. \tag{2}$$

Proposition 9. *Let* $x \neq y$*. Then either* $\mathrm{Orb}(x) = \mathrm{Orb}(y)$*, or* $\mathrm{Orb}(x) \cap \mathrm{Orb}(y) = \varnothing$*.*

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1. GL means "general linear", 2 comes from 2×2, and $\mathbb{R}$ means we take the matrix entries to be real numbers.

2. The key here is to prove that there are no other actions. Mark the balls 1,2,3, and notice that each action must lead to a permutation of $\{1, 2, 3\}$.

Proof. We prove that, if $\mathrm{Orb}(y) \cap \mathrm{Orb}(x) \neq \varnothing$, then the two orbits are equal. To see this, let $z \in \mathrm{Orb}(y) \cap \mathrm{Orb}(x)$. Then there is $g_1, g_2 \in G$ such that $z = g_1\, y$ and $z = g_2\, x$. But then $y = g\, x$ where $g = g_1^{-1}\, g_2$. Now take any $y' \in \mathrm{Orb}(y)$, thus $y' = g'\ y$ for some $g' \in G$. But then $y' = g'\ (g\ x) = (g'\ g)\ x \in \mathrm{Orb}(x)$. Consequently $\mathrm{Orb}(y) \subseteq \mathrm{Orb}(x)$. Now as $x = g^{-1}\, y$, we see that $x \in \mathrm{Orb}(y)$ and repeating the above argument we have $\mathrm{Orb}(x) \subseteq \mathrm{Orb}(y)$, and the two sets must be equal. □

Example 10. Consider the general situation of coloring ball-rod devices with geometric structure. The group G consists of the actions, and the set X is the collection of the m^n devices where the balls are still marked $1, 2, \ldots., n$. Then for $x \in X$, $\mathrm{Orb}(x)$ consists of exactly those devices that are in the same box as x. Proposition 9 now guarantees that the collection of devices can all be boxed.

QUESTION 11. *Let G be a group acting on a set X. Is there an efficient way counting the number of orbits?*

Answer. Burnside's Lemma + Polya's theory of permutation groups.

Burnside's Lemma

LEMMA 12. (BURNSIDE'S LEMMA) *Let G be a group acting on a set X. Then*

$$\text{The number of orbits} = \frac{1}{|G|} \sum_{g \in G} |X_g|, \tag{3}$$

where $|G|$ denotes the number of elements in G, and $|X_g|$ denotes the number of elements in $|X_g|$, with

$$X_g := \{x \in X |\, g\, x = x\} = \{\text{elements in } X \text{ that remains unchanged when acted on by } g\}. \tag{4}$$

Example 13. Consider the 5-ball-4-rods problem. We have seen that $G = \{i,\ f\}$ where i is the identity and f is the flip (rotate by 180 degrees) action. We see that $|G| = 2$. On the other hand, X consists of all possible colorings of 5 balls marked 1,2,3,4,5 by 2 colors.

- Every $x \in X$ is fixed by i. So $|X_i| = 32$.
- $x \in X$ is fixed by f if and only if B1 has the same color as B5, and B2 has the same color as B4. There are 8 of them, that is $|X_f| = 8$.

By Burnside's Lemma, we have

$$\mathrm{Ans} = \frac{32 + 8}{2} = 20. \tag{5}$$

Proof. (OF BURNSIDE'S LEMMA) First we re-write (3) as

$$|G| \times \text{The number of orbits} = \sum_{g \in G} |X_g|. \tag{6}$$

Let's see what the two sides means.

- Left hand sides. We are assuming that every orbit has $|G|$ elements, or equivalently, $g\, x \neq x$ for every $x \in X$. This is clearly an over-count of the elements of X.
- Right hand side. We have

$$\sum_{g \in G} |X_g| = \sum_{g \in G, x \in X, gx = x} 1 = \sum_{x \in X} \left[\sum_{g \in G,\, gx = x} 1 \right]. \tag{7}$$

  This is also clearly an over-count of X as $e\, x = x$ for all x which means each x is counted at least once.

Comparing the two sides, we see that if $g\, x \neq x$ for every $x \in X$, then both sides equals $|X|$. Now we check how many times an arbitrary element in X is counted in the left hand side. Let

$$G_x := \{g \in G |\, g\, x = x\}. \tag{8}$$

We claim that every element in $\mathrm{Orb}(x)$ is counted exactly $|G_x|$ times, that is

$$|\mathrm{Orb}(x)|\,|G_x| = |G|, \tag{9}$$

By definition there are exactly $|G_x|$ elements of G that satisfy $g\,x = x$. Next we notice that if $g'\,x = y \in \mathrm{Orb}(x)$, so does $g'\,g$ for all $g \in G_x$. Therefore there are at least $|G_x|$ elements that takes x to y. On the other hand, fix one particular g' with $g'\,x = y$, then we have $x = (g')^{-1}\,y$ and for every $g''\,x = y$, we have $(g')^{-1}\,g''\,x = x$ so $(g')^{-1}\,g'' \in G_x$. Consequently we see that for every $y \in \mathrm{Orb}(x)$, $|G_y| = |G_x|$. Furthermore we have

$$G_y \cap G_x = \varnothing, \qquad \cup_{y \in \mathrm{Orb}(x)}\, G_y = G. \tag{10}$$

(9) follows.

Next we notice that

$$\begin{aligned}
\sum_{g \in G} |X_g| &= \sum_{g \in G, x \in X, gx = x} 1 \\
&= \sum_{x \in X} \left[\sum_{g \in G, gx = x} 1 \right] \\
&= \sum_{x \in X} |G_x| \\
&= \sum_{\text{Orbits}} |\mathrm{Orb}(x)|\,|G_x| \\
&= \sum_{\text{Orbits}} |G| \\
&= |G| \times \text{The number of orbits.}
\end{aligned} \tag{11}$$

Thus ends the proof. ∎