1 A Few Examples

Example 1. We recall the beads-coloring problem. 5 identical beads are connected by 4 identical rods along a straight line. How many ways are there to color it with two colors R and G?

Solution. First note that the problem is equivalent to: How many different devices of 5 identical beads connected by 4 identical rods, with the beads in color R or G, can be produce?

We do this as follows. First we take many bags of 5 identical beads, number them 1,2,3,4,5, and color them. There are $2^5 = 32$ different ways doing this so we have 32 bags of 5 colored beads. Next we connect then as 1-2-3-4-5, and erase the numbers. During this different bags of beads may turn out to give us the same colored device. We will call two bags "equivalent" if they give us the same colored device. We see that two bags of beads, call them A and B, are equivalent if and only if of A1 has the same color as B5, A2 has the same color as B4, A3 has the same color as B3, A4 has the same color as B2, and A5 has the same color as B1. Thus there are two cases.

i. A bag of beads has the 1st and the 5th beads of the same color, 2nd and 4th of the same color. Each such bag of beads would produce a different colored device.

ii. The remaining bags can be paired up, and each pair produce a different colored device.

As there are 8 bags in the first case, the remaining 24 bags give 12 pairs. Thus the final answer is $8 + 12 = 20$.

Example 2. How many ways are there to color a Merry-Go-Rounds with 10 identical carts located evening along a circle with 2 colors?

Solution. Note that the problem is equivalent to: How many different Merry-Go-Rounds with 10 identical carts in 2 colors (R and G) can we produce?

For this problem it is clear that the relevant actions are to turn the Merry-Go-Rounds through an integer multiple of $\theta_0 := 2\,\pi\,/\,10$, either clockwise or counter-clockwise. We further notice that many of the actions are equivalent. For example, turning clockwise by $99\,\theta_0$ is equivalent to turning clockwise by $9\,\theta_0$ or counter-clockwise by θ_0, while turning couner-clockwise by $20\,\theta_0$ leaves all the carts in their original positions. Therefore there are only 10 relevant actions: Turning clockwise by $0, \theta_0, 2\,\theta_0, \ldots, 9\,\theta_0$. If we number the carts $1, 2, \ldots, 10$ clockwise. Then turning by θ_0 means $1 \longrightarrow 2$, $2 \longrightarrow 3, 3 \longrightarrow 4, \ldots, 10 \longrightarrow 1$.

Now imagine we are producing the Merry-Go-Rounds in the following way. We first build many copies of 10-cart trains, color them, and then connect the first cart with the last. During the last step, many differently colored 10-cart trains will become identical as Merry-Go-Rounds. More precisely, if one colored train becomes another colored train when we disconnect the first train between two carts of the first train, switch their order and connect again, then the two trains will become identical when we turn them into Merry-Go-Rounds. For example,

$$RRRGGRRGRG \Longrightarrow RRRG, \quad GRRGRG \Longrightarrow GRRGRGRRRG \tag{1}$$

implies the two differently colored trains $RRRGGRRGRG$ and $GRRGRGRRRG$ will become identical when we connect the first and the last train and turn them into Merry-Go-Rounds.

We call such pairs of colored trains "equivalent". Note that if one train is equivalent to another train, while the latter train is equivlanet to a third train, then the first train is also equivalent to the third train. Therefore we can group all the colored trains in to groups. Within each group, all trains are equivalent. That is, when connecting the first cart with the last, each group produces exactly one Merry-Go-Rounds. Consequently, all we need to do is to count how many such groups there are.

This could be done if we can figure out how many trains there are in each group. First note that as there are $2^{10} = 1024$ different ways to color a 10-cart train, we have 1024 colored trains.

Take an arbitrary group of trains. As there are 9 possible ways to "cut and reconnect" (cutting between 1st and 2nd, 2nd and 3rd, ...) the maximum number of trains in the group is 10. However the number may be less as some cutting and re-connecting may produce identical trains. There are now 10 cases.

i. The group has 10 trains. We will see at the end that there are 99 of them.

ii. There is a train in the group that becomes itself when cutting between the 1st and the 2nd carts and then re-connect. We claim that in this case there is only one train in the group, colored by one color only. To see this, note that the 1st and the 2nd carts must be of the same color, the 2nd and the 3rd carts must be of the same color, and so on. Thus there are 2 such groups, each has 1 train only.

iii. There is a train in the group that becomes itself when cutting between the 2nd and the 3rd carts and then re-connect. In this case we can break the train into 5 segments of 2 carts each, and clearly all 5 segments must be identical. There are 4 ways of coloring each segments: RR, GG, RG, GR. So here we have 1 more groups of 2 trains.

iv. There is a train in the group that becomes itself when cutting between the 3rd and the 4th carts and then re-connect. This means the 1st cart has the same color as the 4th cart, which has the same color as the 7th cart, which has the same color as the 10th cart, which has the same color as the 3rd cart, which has the same color as the 6th cart, which has the same color as the 9th cart, which has the same color as the 2nd cart, which has the same color as the 5th cart, which has the same color as the 8th cart. Thus all 10 carts must have the same color so this case does not lead to any new groups.

v. There is a train in the group that becomes itself when cutting between the 4th and the 5th carts and then re-connect. This leads to the same groups as in the case of cutting between the 2nd and the 3rd carts.

vi. There is a train in the group that becomes ifself when cutting between the 5th can the 6th carts and then re-connect. We see that trains in this group has the property that front half and the back half are identical. There are 32 different ways to color a 5-cart segment. We can list all of them:

   RRRRR;

   RRRRG, RRRGR, RRGRR, RGRRR, GRRRR,

   RRRGG, RRGGR, RGGRR, GGRRR, GRRRG,

   RRGRG, RGRGR, GRGRR, RGRRG, GRRGR,

The other 16 is obtained through switching R and G. Note that colorings in the same line gives carts in the same group. Thus there 6 new groups of 5 trains each.

vii. The remaining cases do not produce any new groups.

Exercise 1. Prove this!

Summarizing, we have 2 groups of 1 train each, 1 group of 2 trains, and 6 groups of 5 trains. The remaining trains belongs to groups each has 10 trains. Thus there are

$$99 = \frac{1024 - 2 \times 1 - 1 \times 2 - 6 \times 5}{10} \tag{2}$$

groups with 10 trains each. Therefore the number of groups, which is the same as the number of different Merry-Go-Rounds, is

$$99 + 2 + 1 + 6 = 108. \tag{3}$$

Example 3. In Example 2, we notice that Case iii is the same as solving a Merry-Go-Rounds problem of 2 carts, while Case vi is the same as solving a Merry-Go-Rounds problem of 5 carts, and the situation seems to be much simpler than the 10-cart case. For example, in the 5-cart case, we have 2 groups of 1 cart and 6 groups of 5 carts. This is not a coincidence. In general we have the following.

Let p be a prime number. The number of ways of coloring a p-cart Merry-Go-Rounds with m colors is

$$\frac{m^p - m}{p}. \tag{4}$$

Proof. We show that every group of colored trains has p trains, except for m groups of 1 train each (the m trains produced by coloring them with one single color).

It suffice to prove that, if "cutting-and-re-connecting" a train produces an identical train, then the train must be colored by a single color.

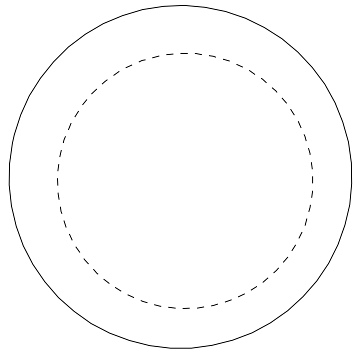
Consider a train that "returns to itself" after cutting between the kth and the $(k+1)$th carts and re-connect. Then we see that the color of the 1st cart is the same as that of the $(k+1)$th cart, which is the same as the lth cart where $l = 2\,k + 1$ or $2\,k + 1 - p$, depending on which one is between 1 and p, and so on. In general, if we let l_i be the remainder of $(i\,k + 1)$ when divided by p (set $l_i = p$ if p divides $i\,k + 1$), then the l_ith cart has the same color as the 1st cart. Now consider the sequence

$$1, k + 1, 2\,k + 1, 3\,k + 1, \ldots., (p - 1)\,k + 1. \tag{5}$$

It is easy to see that when they all have different remainders when divided by p. As there are p numbers, the remainders must be $0, 1, 2, \ldots, p - 1$ (most likely in different order). Therefore all carts has the same color. $\square$

Remark. Note that we have given here a combinatorial proof the Fermat's Little Theorem: $p | (m^p - m)$.

Exercise 2. How many different ways are there to color a Merry-Go-Rounds of the following configuration with three colors?



The two circles are meant to be concentric and the carts are identical.

Exercise 3. How many different ways are there to color a Merry-Go-Rounds with 64 carts using 33 colors? (Hint:[1])

Exercise 4. How many different ways are there to color a Merry-Go-Rounds with n carts with two colors if $n = pq$ where p, q are primes?

1. 64 and 33 are co-prime. Now take a look at https://en.wikipedia.org/wiki/Extended_Euclidean_algorithm.