Recurrence relations

We have seen that the method of ordinary generating function could be used to count the number of ways putting identical balls into boxes (or more generally, the number of integer solutions to equations of the form $a_1 x_1 + \cdots + a_m x_m = n$), while the method of exponential generating function could be used to count the number of ways putting different balls into boxes. For these problems, the generating functions are relatively easy to obtain. On the other hand, there are many other problems for which it is not clear how to get the generating functions. In this lecture we will disucss how to systematically obtain the generating functions when the so-called "recurrence relations", that is a relation between the a_n's, are available. We will do this through examples.

Examples with ordinary generating functions

Example 1. (Fibonacci numbers) The Fibonacci numbers are defined through a recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \qquad n \geqslant 2, \tag{1}$$

with initial conditions $F_0 = F_1 = 1$. Find a general formula for the F_n's.

Solution. We define the ordinary generating function of $\{F_n\}$:

$$F(x) := F_0 + F_1 x + F_2 x^2 + \cdots = \sum_{n=0}^{\infty} F_n x^n. \tag{2}$$

Then we have

$$\begin{aligned} F(x) &= F_0 + F_1 x + \sum_{n=2}^{\infty} F_n x^n \\ &= F_0 + F_1 x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) x^n \\ &= F_0 + F_1 x + \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n \\ &= F_0 + F_1 x + \sum_{n=1}^{\infty} F_n x^{n+1} + \sum_{n=0}^{\infty} F_n x^{n+2} \\ &= F_0 + F_1 x - F_0 x + \sum_{n=0}^{\infty} F_n x^{n+1} + x^2 F(x) \\ &= 1 + (x + x^2) F(x). \end{aligned} \tag{3}$$

Therefore

$$F(x) = \frac{1}{1 - x - x^2}. \tag{4}$$

We see that

$$1 - x - x^2 = 0 \Longrightarrow x = \frac{-1 \pm \sqrt{5}}{2}. \tag{5}$$

Applying partial fraction expansion we have

$$\frac{1}{1 - x - x^2} = \frac{A}{x - \frac{-1 + \sqrt{5}}{2}} + \frac{B}{x - \frac{-1 - \sqrt{5}}{2}}. \tag{6}$$

Multiplying both sides by $\left(x - \frac{-1 + \sqrt{5}}{2}\right)\left(x - \frac{-1 - \sqrt{5}}{2}\right)$, we have

$$1 = A\left(x - \frac{-1 + \sqrt{5}}{2}\right) + B\left(x - \frac{-1 - \sqrt{5}}{2}\right), \tag{7}$$

from which it follows that

$$A = -\frac{1}{\sqrt{5}}, \qquad B = \frac{1}{\sqrt{5}}. \tag{8}$$

Consequently

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n x^n &= F(x) \\
&= \frac{1}{\sqrt{5}}\left[\frac{1}{x - \frac{-1-\sqrt{5}}{2}} - \frac{1}{x - \frac{\sqrt{5}-1}{2}}\right] \\
&= \frac{1}{\sqrt{5}}\left[\frac{1}{\frac{\sqrt{5}-1}{2} - x} - \frac{1}{-\frac{\sqrt{5}+1}{2} - x}\right] \\
&= \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\,\frac{1}{1-\frac{\sqrt{5}+1}{2}x} - \frac{1-\sqrt{5}}{2}\,\frac{1}{1-\frac{1-\sqrt{5}}{2}x}\right] \\
&= \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\sum_{n=0}^{\infty}\left(\frac{\sqrt{5}+1}{2}\right)^n x^n - \frac{1-\sqrt{5}}{2}\sum_{n=0}^{\infty}\left(\frac{1-\sqrt{5}}{2}\right)^n x^n\right] \\
&= \sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]x^n. \qquad (9)
\end{aligned}$$

Consequently

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]. \tag{10}$$

Remark 2. Alternatively, after (6) we know that

$$F_n = A\left(\frac{\sqrt{5}+1}{2}\right)^{n+1} + B\left(\frac{\sqrt{5}-1}{2}\right)^{n+1} \tag{11}$$

for some constants A, B.

Exercise 1. Prove the above claim.

Now setting $n = 0, 1$ we have

$$1 = \frac{\sqrt{5}+1}{2}A + \frac{\sqrt{5}-1}{2}B, \qquad 1 = \left(\frac{\sqrt{5}+1}{2}\right)^2 A + \left(\frac{\sqrt{5}-1}{2}\right)^2 B, \tag{12}$$

from which the values of A, B can be determined.

Exercise 2. The Lucas sequence is defined through $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$. Use generating function to find a formula for L_n. Then calculate $L_1 + L_3 + \cdots + L_{2n-1}$.

Example 3. We have invested 1000 dollars into a savings account that pays five percent interest at the end of each year. At the beginning of each year, we deposit another 500 dollars into this account. How much money will be in this account after n years?

Solution. Let a_n be the amount of money after n years. We have

$$a_0 = 1000, \qquad a_{n+1} = 1.05\, a_n + 500. \tag{13}$$

Now write

$$A(x) := a_0 + a_1 x + a_2 x^n + \cdots \tag{14}$$

We see that

$$\sum_{n\geqslant 0} a_{n+1} x^{n+1} = \sum_{n\geqslant 0} 1.05\, a_n x^{n+1} + \sum_{n\geqslant 0} 500\, x^{n+1}. \tag{15}$$

This leads to

$$A(x) - a_0 = 1.05\, x\, A(x) + \frac{500\, x}{1-x}, \tag{16}$$

from which it follows that

$$A(x) = \frac{1000}{1-1.05\, x} + \frac{500\, x}{(1-x)\,(1-1.05\, x)}. \tag{17}$$

We now apply partial fraction expansion

$$\frac{1}{(1-x)\,(1-1.05\, x)} = \frac{A}{1-x} + \frac{B}{1-1.05\, x}. \tag{18}$$

This gives

$$1 = A\,(1-1.05\, x) + B\,(1-x). \tag{19}$$

Setting $x=1$ we have $A=-20$. Setting $x=0$ we have $A+B=1 \Longrightarrow B=21$. Thus we have

$$\begin{aligned} A(x) &= \frac{1000}{1-1.05\, x} + \frac{500\, x}{(1-x)\,(1-1.05\, x)} \\ &= 1000 \sum_{n=0}^{\infty} 1.05^n x^n \\ &\quad +500\, x \left[\frac{-20}{1-x} + \frac{21}{1-1.05\, x}\right] \\ &= 1000 \sum_{n=0}^{\infty} 1.05^n x^n \\ &\quad -10000\, x \sum_{n=0}^{\infty} x^n + 10500\, x \sum_{n=0}^{\infty} 1.05^n x^n \\ &= 1000 \sum_{n=0}^{\infty} 1.05^n x^n - 10000 \sum_{n=1}^{\infty} x^n + 10500 \sum_{n=1}^{\infty} 1.05^{n-1} x^n \\ &= \sum_{n=0}^{\infty} [1000 \times 1.05^n + 10500 \times 1.05^{n-1} - 10000]\, x^n. \end{aligned} \tag{20}$$

Therefore

$$a_n = 1000 \times 1.05^n + 10500 \times 1.05^{n-1} - 10000 = 11000 \times 1.05^n - 10000. \tag{21}$$

Exercise 3. A bank pays 10% interest at the end of each year on the money in an IRA account at the beginning of the year. Let a_n be the amount of money at the end of year n. Find a formula for a_n. (Ans:[1])

Exercise 4. Show that in a list of all 2^{n-1} compositions of n, $n\geqslant 4$, the integer 3 occurs exactly $n \cdot 2^{n-5}$ times.

Example 4. Let h_n be the maximum number of regions that can be created by n circles in the plane. We have $h_0=1$, $h_1=2$, $h_2=4$, $h_3=8$, $h_4=14$. Find a closed formula for h_n through generating functions.

1. $20000\,(1.1)^n - 20000$.

Solutions. We note that the "best" each new circle could do is to intersect with every existing circle. This leads to

$$h_n = h_{n-1} + 2\,(n-1), \qquad n \geqslant 2 \tag{22}$$

Of course the formula can now be easily obtained. But let's practice the generating function method. Let $H(x) := \sum_{n=0}^{\infty} h_n\, x^n$. Then

$$\begin{aligned} H(x) &= \sum_{n=0}^{\infty} h_n\, x^n \\ &= 1 + 2\,x + \sum_{n=2}^{\infty} h_n\, x^n \\ &= 1 + 2\,x + \sum_{n=2}^{\infty} (h_{n-1} + 2\,(n-1))\, x^n \\ &= 1 + 2\,x + \sum_{n=2}^{\infty} h_{n-1}\, x^n + 2 \sum_{n=2}^{\infty} (n-1)\, x^n \\ &= 1 + 2\,x + \sum_{n=1}^{\infty} h_n\, x^{n+1} + 2 \sum_{n=1}^{\infty} n\, x^{n+1} \\ &= 1 + 2\,x + \left[x \sum_{n=0}^{\infty} h_n\, x^n - x \right] + 2\,x^2 \sum_{n=0}^{\infty} (x^n)' \\ &= 1 + x + x\, H(x) + \frac{2\,x^2}{(1-x)^2}. \end{aligned} \tag{23}$$

Therefore

$$H(x) = \frac{1+x}{1-x} + \frac{2\,x^2}{(1-x)^3} = (1+x) \sum_{n=0}^{\infty} x^n + x^2 \sum_{n=0}^{\infty} (n+2)\,(n+1)\, x^n. \tag{24}$$

Therefore

$$h_n = 2 + n\,(n-1). \tag{25}$$

Example 5. Let $a_{n+2} = 3\, a_{n+1} - 2\, a_n$ for $n \geqslant 0$, and let $a_0 = 0$, $a_1 = 1$. Find an explicit formula for a_n.

Solution. Note that the recurrence relation can be written as

$$a_n = 3\, a_{n-1} - 2\, a_{n-2}, \qquad n \geqslant 2. \tag{26}$$

Setting $A(x) = \sum_{n=0}^{\infty} a_n\, x^n$ we have

$$\begin{aligned} A(x) &= a_0 + a_1\, x + \sum_{n=2}^{\infty} a_n\, x^n \\ &= x + \sum_{n=2}^{\infty} (3\, a_{n-1} - 2\, a_{n-2})\, x^n \\ &= x + 3\, x \sum_{n=2}^{\infty} a_{n-1}\, x^{n-1} - 2\, x^2 \sum_{n=2}^{\infty} a_{n-2}\, x^{n-2} \\ &= x + 3\, x \sum_{n=1}^{\infty} a_n\, x^n - 2\, x^2 \sum_{n=0}^{\infty} a_n\, x^n \\ &= x + 3\, x \sum_{n=0}^{\infty} a_n\, x^n - 2\, x^2 \sum_{n=0}^{\infty} a_n\, x^n \\ &= x + 3\, x\, A(x) - 2\, x^2\, A(x). \end{aligned} \tag{27}$$

Thus

$$A(x) = \frac{x}{1 - 3x + 2x^2}, \tag{28}$$

from which we can apply partial fraction expansion to obtain $a_n = 2^n - 1$.

In general we have the following theorem.

Theorem 6. *Consider the recurrence relation*

$$h_n = a_1 h_{n-1} + \cdots + a_k h_{n-k}, \qquad n \geqslant k, a_k \neq 0. \tag{29}$$

Assume that the polynomial

$$x^k - a_1 x^{k-1} - \cdots - a_k = 0 \tag{30}$$

has k distinct roots $q_1, \ldots, q_k$, then

$$h_n = c_1 q_1^n + \cdots + c_k q_k^n \tag{31}$$

for some constants $c_1, \ldots, c_k$ determined though the values of $h_0, \ldots, h_{k-1}$.

We will not prove the theorem here but leave it as an exercise for anyone interested in theory.

Exercise 5. What happens if we have repeated roots? (Hint:[2])

It is also possible to sove recurrence relations that are not homogeneous.

Example 7. Solve $h_n = 2 h_{n-1} + 3^n$, $n \geqslant 1$ and $h_0 = 2$.

Solution. Let $H(x) = \sum_{n=0}^{\infty} h_n x^n$. We have

$$\begin{aligned} H(x) &= h_0 + \sum_{n=1}^{\infty} h_n x^n \\ &= 2 + \sum_{n=1}^{\infty} (2 h_{n-1} + 3^n) x^n \\ &= 2 + 2 \sum_{n=1}^{\infty} h_{n-1} x^n + \sum_{n=1}^{\infty} 3^n x^n \\ &= 2 + 2 \sum_{n=0}^{\infty} h_n x^{n+1} + \frac{3x}{1 - 3x} \\ &= 2x H(x) + \frac{2 - 3x}{1 - 3x}. \end{aligned} \tag{32}$$

Therefore

$$H(x) = \frac{2 - 3x}{(1 - 2x)(1 - 3x)}. \tag{33}$$

Application of partial fraction expansion gives

$$H(x) = \frac{-1}{1 - 2x} + \frac{3}{1 - 3x}. \tag{34}$$

2. What do we do in partial fraction expansion when there are repeated roots?

Consequently $h_n = -2^n + 3^{n+1}$.

Exercise 6. Solve

$$h_n = h_{n-1} + n^3, \qquad h_0 = 0. \tag{35}$$

(Ans: [3])

Exercise 7. Solve

$$h_n = 3\,h_{n-1} + 3^n, \qquad h_0 = 2. \tag{36}$$

(Ans:[4])

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3. $\frac{n^2\,(n+1)^2}{4}$.
4. $(2+n)\,3^n$.