Integer solutions

Example 1. If 25 identical juggling balls are distributed to five different jugglers, with each juggler receiving at least 3 juggling balls, how many distributions are possible?

Solution. The generating functions is

$$(x^3+x^4+\cdots)^5=\frac{x^{15}}{(1-x)^5}=x^{15}\sum_{n=0}^{\infty}\frac{1}{4!}(n+4)(n+3)(n+2)(n+1)x^n. \tag{1}$$

The coefficient for x^{14} is then

$$\frac{14\times 13\times 12\times 11}{4!}=C(14,4). \tag{2}$$

Remark 2. We see that the problem is equivalent to finding the number of integer solutions to

$$x_1+x_2+x_3+x_4+x_5=25, \qquad x_i\geqslant 3. \tag{3}$$

Example 3. Count the number of selections of 30 toys from 10 different types of toys if at least two of each kind must be selected.

Solution. The generating function is

$$(x^2+x^3+\cdots)^{10}=x^{20}\frac{1}{(1-x)^{10}}=\frac{x^{20}}{9!}\sum_{n=0}^{\infty}(n+9)\cdots(n+1)x^n. \tag{4}$$

The coefficient of x^{30} is then given by

$$\frac{19\times\cdots\times 11}{9!}=\binom{19}{9}. \tag{5}$$

Exercise 1. Write down the equivalent "integer solutions" problem to Example 3.

Example 4. An elementary school class consisting of one teacher and 25 students donates 20 dollars to a local charity. In how many ways can this be done if the teacher donates 0, 2, or 4 dolloars and each student donates 0 or 1 dollar?

Solution. The generating function is

$$(1+x^2+x^4)(1+x)^{25}. \tag{6}$$

The answer is $C(25,20)+C(25,18)+C(25,16)$.

Exercise 2. In how many ways can a charity collect 20 dollars from 12 children and two adults if each child gives one or two dollars and each adult gives from one to five dolloars?

Example 5. There are 50 identical weights of 1g each and 40 identical weights of 2g each. How many different ways are there to obtain 60g from these weights?

Solution. The answer is given by the coefficent of x^{60} in the expansion of

$$(1+x+\cdots+x^{50})(1+x^2+\cdots+x^{80}). \tag{7}$$

We re-write as

$$(1-x^{51})(1-x^{82})\left[\frac{1}{1-x}\frac{1}{1-x^2}\right] \tag{8}$$

The idea now is to apply the method of "partial fractions" to turn the product $\frac{1}{1-x}\frac{1}{1-x^2}$ into a sum of similar terms.

We write

$$\begin{aligned}\frac{1}{1-x}\frac{1}{1-x^2} &= \frac{1}{(1-x)^2}\frac{1}{1+x}\\ &= \frac{A}{(1-x)}+\frac{B}{(1-x)^2}+\frac{C}{1+x}.\end{aligned}$$

Multiply both sides by $(1-x)^2(1+x)$ we have

$$1=A(1-x)(1+x)+B(1+x)+C(1-x)^2. \tag{9}$$

Setting $x=1$ we have $B=1/2$. Setting $x=-1$ we have $C=1/4$. Taking derivative and then setting $x=1$ we have $A=1/4$. Thus we have

$$\begin{aligned}\frac{1}{1-x}\frac{1}{1-x^2} &= \frac{1/4}{1-x}+\frac{1/2}{(1-x)^2}+\frac{1/4}{1+x}\\ &= \frac{1}{4}\sum_{n=0}^{\infty}x^n+\frac{1}{2}\sum_{n=0}^{\infty}(n+1)x^n+\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n\\ &= \sum_{n=0}^{\infty}\left[\frac{n}{2}+\frac{3+(-1)^n}{4}\right]x^n. \end{aligned}\tag{10}$$

Finally, the answer is given by

$$\left[\frac{n}{2}+\frac{3+(-1)^n}{4}\right]_{n=60}-\left[\frac{n}{2}+\frac{3+(-1)^n}{4}\right]_{n=9}=26. \tag{11}$$

Example 6. Find the number of integers whose digits sum to 23, among integers from 0 to 9999.

Solution. The generating function for this problem is

$$\begin{aligned}(1+\cdots+x^9)^4 &= (1-x^{10})^4\frac{1}{(1-x)^4}\\ &= (1-x^{10})^4\frac{1}{3!}\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)x^n\\ &= \left(1-\binom{4}{1}x^{10}+\binom{4}{2}x^{20}-\binom{4}{3}x^{30}+\binom{4}{4}x^{40}\right)\cdot\\ &\quad \frac{1}{3!}\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)x^n. \end{aligned}\tag{12}$$

The answer is the coefficient for x^{23} in the expansion which is given by

$$\binom{26}{3}-4\cdot\binom{16}{3}+6\cdot\binom{6}{3}=2600-2240+120=480. \tag{13}$$

Exercise 3. Find the number of four-digit integers whose digits sum to 23. (Hint:[1])

Exercise 4. Solve Example 6 by inclusion-exclusion.

Exercise 5. Use generating function method to find the number of solutions to

$$x_1+x_2+x_3+x_4+x_5=15, \qquad 0\leqslant x_i\leqslant 5. \tag{14}$$

Exercise 6. Find a generating function for each of the following.

a) 5 distinct dice with a sum of n.

b) 6 distinct dice with a sum of n, and the ith die does not show the value i.

Partitions

Recall that a partition of n into m summands is a distribution of n identical balls into m identical boxes with no box empty. We denote the number of ways doing this by $p_m(n)$.

Thus $p_m(n)$ is the same as the number of integer solutions to

$$x_1+\cdots+x_m=n, \qquad x_1\geqslant x_2\geqslant\cdots\geqslant x_m\geqslant 1. \tag{15}$$

If we denote

$$y_m=x_m-1, \quad y_{m-1}=x_{m-1}-x_m, \ldots, y_1=x_1-x_2, \tag{16}$$

1. $(x+\cdots+x^9)(1+x+\cdots+x^9)^3$

we have

$$x_m = y_m + 1, \quad x_{m-1} = y_{m-1} + y_m + 1, \quad \cdots \quad x_1 = y_1 + \cdots + y_m + 1. \tag{17}$$

Consequently

$$y_1 + 2\,y_2 + \cdots + m\,y_m = n - m, \qquad y_i \geqslant 0. \tag{18}$$

Thus we see that for fixed m, the generating function for $p_m(n)$ is

$$\begin{aligned} P_m(x) &= x^m\,(1 + x + x^2 + \cdots) \cdots (1 + x^m + x^{2m} + \cdots) \\ &= \frac{x^m}{(1-x)\,(1-x^2) \cdots (1-x^m)}. \end{aligned} \tag{19}$$

Exercise 7. Obtain (19) through the following observation instead of the theory of integer solutions.

The number of different ways to partition n into m summands is the same as the number of different ways to partition n into summands with the largest summand equal to m.

Now if we allow the boxes to be empty, that is we partition n into no more than m summands, then the generating function is

$$\frac{1}{(1-x)\,(1-x^2) \cdots (1-x^m)}. \tag{20}$$

Finally, if we do not put any restriction on how many boxes we are distributing the balls into, the generating function for the number of different ways, $p(n)$, is

$$\prod_{m=1}^{\infty} \frac{1}{1-x^m} := \frac{1}{(1-x)\,(1-x^2) \cdots (1-x^m) \cdots}. \tag{21}$$

We have seen that, although this infinite product looks complicated, it can be conveniently utilized to prove nontrivial results.

Example 7. Find a generating function for a_n, the number of different triangles with integral sides and perimeter n.

Solution. We see that this is the same as the number of integer solutions to

$$x_1 + x_2 + x_3 = n, \qquad x_1 \geqslant x_2 \geqslant x_3 > 0, \qquad x_1 < x_2 + x_3. \tag{22}$$

Now let $y_3 = x_3 - 1$, $y_2 = x_2 - x_3$, $y_1 = x_1 - x_2$. We see that

$$y_1 + 2\,y_2 + 3\,y_3 = n - 3, \qquad y_1, y_2, y_3 \geqslant 0, \qquad y_1 \leqslant y_3. \tag{23}$$

Now let $u_1 = y_1$, $u_2 = y_2$, $u_3 = y_3 - y_1$, we see that

$$4\,u_1 + 2\,u_2 + 3\,u_3 = n - 3, \qquad u_i \geqslant 0. \tag{24}$$

Therefore the generating function is

$$x^3\,(1 + x^4 + x^8 + \cdots)\,(1 + x^2 + x^4 + \cdots)\,(1 + x^3 + x^6 + \cdots) \tag{25}$$

which reduces to

$$\frac{x^3}{(1-x^4)\,(1-x^2)\,(1-x^3)}. \tag{26}$$