Ordinary generating functions

DEFINITION 1. (ORDINARY GENERATING FUNCTION) *Let $a_0, a_1, \ldots$ be a sequence of numbers. The power series $A(x) := a_0 + a_1 x + a_2 x^2 + \cdots$ is called the "generating function" of the sequence.*

NOTATION 2. *It is convenient to use the shorthand $\sum_{n=0}^{\infty} a_n x^n$ to denote the power series $a_0 + a_1 x + \cdots$. Note that $\sum_{n=0}^{\infty} a_n x^n$ is just another way of writing $a_0 + a_1 x + \cdots$, nothing more.*

Remark 3. When there are only finitely many a_n's, the generating function of the sequence is a polynomial. On the other hand, for practical purposes, a "power series" can be treated as a "polynomial of infinite degree"[1]. Thus we naturally have the following rules for operations of power series.

Operations of power series

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n. \tag{1}$$

$$c \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (c\, a_n) x^n. \tag{2}$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n. \tag{3}$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n. \tag{4}$$

$$\int_0^x \left(\sum_{n=0}^{\infty} a_n z^n\right) \mathrm{d}z = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n. \tag{5}$$

Remark. It is crucial to understand that the index n in the power series $\sum_{n=0}^{\infty} a_n x^n$ is only a "place holder". It's whole purpose is to indicate that the subscript of the coefficient and the power of x are the same, and that the sum starts from the zeroth term. Therefore we can replace n by any other symbol:

$$\sum_{n=0}^{\infty} a_n x^n, \quad \sum_{m=0}^{\infty} a_m x^m, \quad \sum_{k=0}^{\infty} a_k x^k \tag{6}$$

all denote the **same** power series

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \tag{7}$$

However, they are not the same as

$$\sum_{n=2}^{\infty} a_n x^n \text{ or } \sum_{k=0}^{\infty} a_k x^{k+1} \tag{8}$$

as the former starts from a different term, and the latter has a different relation between the subscript and the power.

Example 4. Let $A(x) = 1 + x^2 + 3x^5$ and $B(x) = 4 + x + 2x^3 + x^5$.

a) Compute $A(x) + B(x)$;

b) Compute $A(x)\, B(x)$.

Solution.

a) We have

$$A(x) = 1 \cdot x^0 + 0 \cdot x^1 + 1 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 + 3 \cdot x^5 \tag{9}$$

and

$$B(x) = 4 \cdot x^0 + 1 \cdot x^1 + 0 \cdot x^2 + 2 \cdot x^3 + 0 \cdot x^4 + 1 \cdot x^5 \tag{10}$$

1. This is what Newton did!

so

$$\begin{aligned} A(x)+B(x) &= (1+4)\cdot x^0+(0+1)\cdot x^1+(1+0)\cdot x^2 \\ &\quad +(0+2)\cdot x^3+(0+0)\cdot x^4+(3+1)\cdot x^5 \\ &= 5+x+x^2+2\,x^3+4\,x^5. \end{aligned} \tag{11}$$

b) By (9,10) we have

$$\begin{aligned} A(x)\,B(x) &= (1\times 4)\,x^0+(1\times 1+0\times 4)\,x^1 \\ &\quad +(1\times 0+0\times 1+1\times 4)\,x^2+\cdots \\ &\quad +(3\times 1)\,x^{10} \\ &= 4+x+4\,x^2+3\,x^3+15\,x^5+3\,x^6+x^7+6\,x^8+3\,x^{10}. \end{aligned} \tag{12}$$

Example 5. Let $A(x):=2+3\,x+4\,x^2+\cdots$ and $B(x):=1+3\,x+5\,x^2+\cdots$.

a) Write $A(x), B(x)$ into the compact form.

b) Calculate $A(x)+B(x)$.

c) Calculate $A(x)\,B(x)$.

d) Calculate $A'(x)$.

Solution.

a) We have

$$A(x)=\sum_{n=0}^{\infty}(n+2)\,x^n, \qquad B(x)=\sum_{n=0}^{\infty}(2\,n+1)\,x^n. \tag{13}$$

b) We have

$$A(x)+B(x)=\sum_{n=0}^{\infty}3\,(n+1)\,x^n. \tag{14}$$

c) We have the coefficient of x^n in $A(x)\,B(x)$ to be

$$\begin{aligned} \sum_{k=0}^{n}(k+2)\,(2\,(n-k)+1) &= \sum_{k=0}^{n}(k+2)\,[(2\,n+1)-2\,k] \\ &= \sum_{k=0}^{n}[2\,(2\,n+1)+(2\,n-3)\,k-2\,k^2] \\ &= 2\,(n+1)\,(2\,n+1)+(2\,n-3)\sum_{k=0}^{n}k \\ &\quad -2\sum_{k=0}^{n}k^2 \\ &= 2\,(n+2)\,(2\,n+1)+(2\,n-3)\,\frac{n\,(n+1)}{2} \\ &\quad -2\,\frac{n\,(n+1)\,(2\,n+1)}{6} \\ &= \frac{n^3}{3}+\frac{5}{2}\,n^2+\frac{25\,n}{6}+2. \end{aligned} \tag{15}$$

d) We have

$$\begin{aligned} A'(x) &= \left(\sum_{n=0}^{\infty}(n+2)\,x^n\right)' \\ &= \sum_{n=1}^{\infty}(n+2)\,n\,x^{n-1} \\ &= \sum_{n=0}^{\infty}(n+3)\,(n+1)\,x^n. \end{aligned} \tag{16}$$

Taylor expansion

In Combinatorics we usually do the Taylor expansion at 0.

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}\,x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\,x^n + \cdots \tag{17}$$

In essence, Taylor expansion is the following relation

$$f(x) = \text{a power series} = \text{a polynomial of degree infinity}. \tag{18}$$

The most useful Taylor expansion are

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n, \tag{19}$$

and

$$e^x = 1 + x + \frac{x^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{20}$$

Note that from (19) we have

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \sum_{n=1}^{\infty} n\,x^{n-1} = \sum_{n=0}^{\infty} (n+1)\,x^n, \tag{21}$$

$$\frac{2}{(1-x)^3} = \left(\frac{1}{(1-x)^2}\right)' = \sum_{n=1}^{\infty} (n+1)\,n\,x^{n-1} = \sum_{n=0}^{\infty} (n+2)\,(n+1)\,x^n. \tag{22}$$

and so on.

Example 6. Let $A(x) := 2 + 3\,x + 4\,x^2 + \cdots$ and $B(x) := 1 + 3\,x + 5\,x^2 + \cdots$. Calculate $A(x)\,B(x)$.

Solution. We recall

$$A(x) = \sum_{n=0}^{\infty} (n+2)\,x^n, \qquad B(x) = \sum_{n=0}^{\infty} (2\,n+1)\,x^n. \tag{23}$$

Therefore

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (n+1)\,x^n \\ &= \frac{1}{1-x} + \frac{1}{(1-x)^2}. \end{aligned} \tag{24}$$

and

$$\begin{aligned} B(x) &= 2\sum_{n=0}^{\infty} (n+1)\,x^n - \sum_{n=0}^{\infty} x^n \\ &= \frac{2}{(1-x)^2} - \frac{1}{1-x}. \end{aligned} \tag{25}$$

Therefore

$$\begin{aligned} A(x)\,B(x) &= \frac{2}{(1-x)^4} + \frac{1}{(1-x)^3} - \frac{1}{(1-x)^2} \\ &= \frac{1}{3}\left(\frac{1}{1-x}\right)''' + \frac{1}{2}\left(\frac{1}{1-x}\right)'' - \left(\frac{1}{1-x}\right)' \\ &= \frac{1}{3}\sum_{n=0}^{\infty} (n+3)\,(n+2)\,(n+1)\,x^n \\ &\quad + \frac{1}{2}\sum_{n=0}^{\infty} (n+2)\,(n+1)\,x^n - \sum_{n=0}^{\infty} (n+1)\,x^n \\ &= \frac{n^3}{3} + \frac{5}{2}n^2 + \frac{25\,n}{6} + 2. \end{aligned}$$

The method of partial fractions

The basic idea is to write $\frac{P}{Q}$, where P, Q are polynomials with degree of P less than degree of Q, into the sum of functions of the type $\frac{A}{(s-r)^m}$. It is done through the following steps.

1. Factorize Q:
$$Q(s) = (s - r_1) \cdots (s - r_n). \tag{26}$$

2. Go through $r_1, \ldots, r_n$ and write down the terms of the RHS sum of
$$\frac{P}{Q} = \sum \cdots \tag{27}$$
according to the following rules:

   i. If r_i is a single real root, write down
$$\frac{A_i}{s - r_i}. \tag{28}$$

   ii. If r_i is a repeated real root, say with multiplicity m, write down
$$\frac{A_{i1}}{s - r_i} + \frac{A_{i2}}{(s - r_i)^2} + \cdots + \frac{A_{im}}{(s - r_i)^m}. \tag{29}$$
After this, discard those other copies of r_i from the list $r_1, \ldots, r_n$ and move on to the next root. Note that the previous "single root" case is actually contained in this case.

   iii. If $r_i = \alpha + i\,\beta$ is complex root with multiplicity m, then there must be another $r_j = \alpha - i\,\beta$ with the same multiplicity. Write down
$$\frac{C_{i1}\,s + D_{i1}}{(s - \alpha)^2 + \beta^2} + \cdots + \frac{C_{im}\,s + D_{im}}{[(s - \alpha)^2 + \beta^2]^m}. \tag{30}$$

   For example, if
$$Q(s) = (s - 1)(s - 3)^3\,(s + i)\,(s - i), \tag{31}$$
we have six roots (counting multiplicity) $1, 3, 3, 3, -i, i$. Now to form the RHS, we go through this list one by one:
$$1\text{: Single real root} \Longrightarrow \frac{A}{s-1}; \tag{32}$$
$$3\text{: repeated real root with multiplicity 3} \Longrightarrow \frac{B}{s-3} + \frac{C}{(s-3)^2} + \frac{D}{(s-3)^3}; \tag{33}$$
$$\text{Ignore the remaining two 3's.} \tag{34}$$
$$-i\text{: Complex root with multiplicity 1} \Longrightarrow \frac{E\,s + F}{s^2 + 1}; \tag{35}$$
$$\text{Ignore the complex conjugate } i. \tag{36}$$

3. Determine the constants using the following procedure: We use the above example
$$Q(s) = (s - 1)(s - 3)^3\,(s + i)\,(s - i), \tag{37}$$
which gives
$$\frac{P}{Q} = \frac{A}{s-1} + \frac{B}{s-3} + \frac{C}{(s-3)^2} + \frac{D}{(s-3)^3} + \frac{E\,s + F}{s^2 + 1} \tag{38}$$
leading to
$$P(s) = A\,(s-3)^3\,(s^2+1) + B\,(s-1)\,(s-3)^2\,(s^2+1) + C\,(s-1)\,(s-3)\,(s^2+1) + D\,(s-1)\,(s^2+1) + (E\,s + F)\,(s-1)\,(s-3)^3. \tag{39}$$

   i. Set s to be each of the single real roots. This would immediately give all the constants corresponding to those single roots.

      In our example, we see that setting $s = 1$ immediately gives A.

ii. Set s to be the repeated real roots. This would immediately give all the constants in the last terms of the terms corresponding to those repeated roots.

In our example, setting $s=3$ immediately gives D.

- At this stage, you may want to try the "differentiation method". In our example, differentiating once we obtain

$$\begin{aligned} P'(s) &= A\,[2\,(s-3)\,(s^2+1)+(s-3)^2\,(2\,s)] \\ &\quad +B\,[(s-3)^2\,(s^2+1)+2\,(s-1)\,(s-3)\,(s^2+1)+2\,s\,(s-1)\,(s-3)^2] \\ &\quad +C\,[(s-3)\,(s^2+1)+(s-1)\,(s^2+1)+2\,s\,(s-1)\,(s-3)] \\ &\quad +D\,[s^2+1+2\,s\,(s-1)] \\ &\quad +E\,[(s-1)\,(s-3)^3]+(E\,s+F)\,[(s-3)^3+3\,(s-1)\,(s-3)^2]. \end{aligned} \tag{40}$$

  Looks very complicated, but as soon as we substitute $s=3$, only C and D remain. As we have already found D, determining C is easy.

  Differentiate again and then set $s=3$, we obtain one equation for B, C, D. Since we already know C, D, B is immediately determined.

iii. Set $s=0$.

iv. If there are still some constants need to be determined, compare the coefficient for the highest power term s^n of the RHS. Note that as P has lower degree, we always have $0=\cdots$. In our example,

$$P(s) = A\,(s-3)^3\,(s^2+1)+B\,(s-1)\,(s-3)^2\,(s^2+1)+C\,(s-1)\,(s-3)\,(s^2+1)+D\,(s-1)\,(s^2+1)+(E\,s+F)\,(s-1)\,(s-3)^3. \tag{41}$$

The higher order term on the RHS is s^5. Assuming

$$P(s)=p_5\,s^5+\cdots \tag{42}$$

we have

$$p_5=A+B+E. \tag{43}$$

Note that this is equivalent to setting $s=\infty$.

v. Let's say there are k constants still need to be determined. Set s to be k arbitrary values. You will obtain k equations for these k costants, solve them.

In our example, $k=0$ if we have used the "differentiation method", $k=2$ if we haven't.

Example 7. Compute the partial fraction expansion of

$$\frac{6\,s^2-13\,s+2}{s\,(s-1)\,(s-6)}. \tag{44}$$

Solution. First we check that the degree of the denominator is indeed higher than the degree of the nominator. Thus we can write

$$\frac{6\,s^2-13\,s+2}{s\,(s-1)\,(s-6)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-6}. \tag{45}$$

Summing the RHS gives

$$\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-6} = \frac{A\,(s-1)\,(s-6)+B\,s\,(s-6)+C\,s\,(s-1)}{s\,(s-1)\,(s-6)} \tag{46}$$

We need to find A, B, C such that

$$A\,(s-1)\,(s-6)+B\,s\,(s-6)+C\,s\,(s-1)=6\,s^2-13\,s+2. \tag{47}$$

Naïvely, one may want to expand the LHS into

$$(A+B+C)\,s^2+(-7\,A-6\,B-C)\,s+6\,A \tag{48}$$

and then solve

$$
\begin{aligned}
A+B+C &= 6 && (49)\\
-7A-6B-C &= -13 && (50)\\
6A &= 2. && (51)
\end{aligned}
$$

However there is a much simpler way. The key observation is that when we set $s=0,1,6$, exactly two of the three terms vanish. In other words, when we set $s=0,1,6$, exactly one unknown is left in the equation – one equation, one unknown, linear: the simplest equation possible!

- Setting $s=0$, we have
$$A\,(0-1)\,(0-6)=2 \Longrightarrow A=1/3. \tag{52}$$
- Setting $s=1$, we have
$$B\,(1-6)=-5 \Longrightarrow B=1. \tag{53}$$
- Setting $s=6$, we have
$$C\,6\,(6-1)=216-78+2=140 \Longrightarrow C=14/3. \tag{54}$$

Thus the solution is

$$A=\frac{1}{3}, \quad B=1, \quad C=\frac{14}{3}. \tag{55}$$

Example 8. Compute the partial fraction expansion of

$$\frac{5\,s^2+34\,s+53}{(s+3)^2\,(s+1)}. \tag{56}$$

Solution. Again, we first check that the nominator's degree is lower.

Next we write the function into partial fractions:

$$\frac{5\,s^2+34\,s+53}{(s+3)^2\,(s+1)}=\frac{A}{s+3}+\frac{B}{(s+3)^2}+\frac{C}{s+1}. \tag{57}$$

Calculating the RHS, we have

$$\frac{A}{s+3}+\frac{B}{(s+3)^2}+\frac{C}{s+1} = \frac{A\,(s+3)\,(s+1)+B\,(s+1)+C\,(s+3)^2}{(s+3)^2\,(s+1)}. \tag{58}$$

We need A, B, C such that

$$A\,(s+3)\,(s+1)+B\,(s+1)+C\,(s+3)^2=5\,s^2+34\,s+53. \tag{59}$$

Setting $s=-3$, we have

$$B\,(-3+1)=45-102+53=-4 \Longrightarrow B=2. \tag{60}$$

Setting $s=-1$, we have

$$C\,(-1+3)^2=5-34+53=24 \Longrightarrow C=6. \tag{61}$$

To determine A, we pick $s=0$ to obtain

$$3\,A+B+9\,C=53 \Longrightarrow A=-1. \tag{62}$$

Example 9. Compute the partial fraction expansion of

$$\frac{7\,s^2+23\,s+30}{(s-2)\,(s^2+2\,s+5)}. \tag{63}$$

Solution. Again, the degree of the nominator is lower. Check.

We write

$$\frac{7\,s^2+23\,s+30}{(s-2)\,(s^2+2\,s+5)}=\frac{A}{s-2}+\frac{B\,s+C}{s^2+2\,s+5}=\frac{A\,(s^2+2\,s+5)+(B\,s+C)\,(s-2)}{(s-2)\,(s^2+2\,s+5)}. \tag{64}$$

We need to find A, B, C such that

$$A\,(s^2+2\,s+5)+(B\,s+C)\,(s-2)=7\,s^2+23\,s+30. \tag{65}$$

Setting $s=2$ we have

$$A\,(4+4+5)=28+46+30=104 \Longrightarrow A=8. \tag{66}$$

To find B, C, we need to set s to values different from 2 and obtain equations for B, C. There is a minor trick here that can make the equations simple. We notice that the B disappears if we set $s=0$. Setting $s=0$ we have

$$5\,A-2\,C=30 \Longrightarrow C=5. \tag{67}$$

Finally comparing the s^2 terms (or setting s to yet another value) we have

$$A+B=7 \Longrightarrow B=-1. \tag{68}$$