1. Integer solutions.

- Basic cases.
  - $x_1+\cdots+x_m=n, x_i>0$. $C(n-1, m-1)$.
  - $x_1+\cdots+x_m=n, x_i\geqslant 0$. $C(n+m-1, m-1)$.
  - $x_1+\cdots+x_m=n$, $x_i\geqslant a_i$. $C(n-a_1-\cdots-a_m+m-1, m-1)$.

  Note. Sometimes the natural set-up of a problem gives

$$x_1+\cdots+x_m=n, \qquad x_i>0 \text{ for some } i's, \geqslant 0 \text{ for other } i's. \tag{1}$$

  The above formulas do not **directly** apply here.

- More complicated cases.

$$x_1+\cdots+x_m=n, \qquad a_i\leqslant x_i<b_i. \tag{2}$$

  Use inclusion-exclusion.

  1. First let

$$A_0=\text{solutions to } x_1+\cdots+x_m=n, a_i\leqslant x_i. \tag{3}$$

  2. Our goal is to identify those solutions in A_0 that satisfy furthermore $x_1<b_1, x_2<b_2, \ldots, x_m<b_m$.

     This is equivalent to identification of those solutions in A_0 with at least one of the following properties $P_1, \ldots, P_m$ where

$$P_i\text{: the soluiont violates } x_i<b_i. \tag{4}$$

  3. If we let A_i be the set of solutions in A_0 with property P_i, that is

$$A_i=\text{solutions to } x_1+\cdots+x_m=n, x_i\geqslant b_i, x_j\geqslant a_j \text{ for other } j, \tag{5}$$

     then the answer to the original problem is given by

$$|A_0|-|A_1\cup\cdots\cup A_m| \tag{6}$$

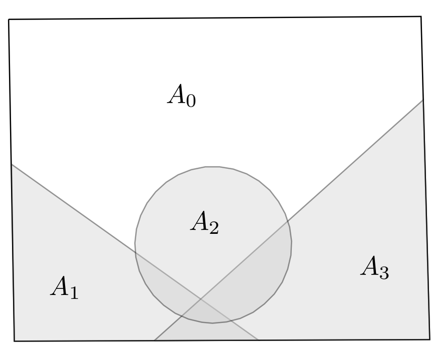

Figure 1. Inclusion-exclusion for integer solutions problems

4. We calculate $|A_1 \cup \cdots \cup A_m|$ through inclusion-exclusion:

$$
\begin{aligned}
|A_1 \cup \cdots \cup A_n| &= \sum_{i=1}^{n} |A_i| \\
&\quad - \sum_{i,j=1, i\neq j}^{n} |A_i \cap A_j| \\
&\quad + \sum_{i,j,k=1, i,j,k \text{ distinct}}^{n} |A_i \cap A_j \cap A_k| \\
&\quad - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|.
\end{aligned} \tag{7}
$$

Example 1. How many terms are there in the expansion of $(x+y+z)^7$?

Solution. Each term is of the form $x^a\, y^b\, z^c$. Thus the answer is the same as the number of integer solutions to

$$a+b+c=7, \qquad a, b, c \geqslant 0 \tag{8}$$

which is $\binom{9}{2} = 36$.

Example 2. How many non-negative integer solutions are there for the inequality $x_1 + \cdots + x_m \leqslant n$?

Solution. Introduce $x_{m+1} = n - x_1 - \cdots - x_m$. Then we see that the answer is $\binom{n+m}{m}$.

Example 3. How many solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 = 20 \tag{9}$$

in positive integers with $x_1 \leqslant 6, x_2 \leqslant 7, x_3 \leqslant 8$, and $x_4 \leqslant 9$?

Solution. Let

- A_0=solutions to $x_1 + x_2 + x_3 + x_4 = 20,\ x_i > 0$;
- A_1=solutions to $x_1 + x_2 + x_3 + x_4 = 20,\ x_1 > 6, x_2, x_3, x_4 > 0$;
- A_2 = solutions to $x_1 + x_2 + x_3 + x_4 = 20,\ x_2 > 7, x_1, x_3, x_4 > 0$;
- A_3 = solutions to $x_1 + x_2 + x_3 + x_4 = 20,\ x_3 > 8, x_1, x_2, x_4 > 0$;
- A_4 = solutions to $x_1 + x_2 + x_3 + x_4 = 20,\ x_4 > 9,\ x_1, x_2, x_3 > 0$.

We calculate

$$
\begin{aligned}
|A_0| &= \binom{19}{3} = 969; \\
|A_1| &= \binom{13}{3} = 286; \\
|A_2| &= \binom{12}{3} = 220; \\
|A_3| &= \binom{11}{3} = 132; \\
|A_4| &= \binom{10}{3} = 120; \\
|A_1 \cap A_2| &= \binom{6}{3} = 20; \\
|A_1 \cap A_3| &= \binom{5}{3} = 10; \\
|A_1 \cap A_4| &= \binom{4}{3} = 6; \\
|A_2 \cap A_3| &= \binom{4}{3} = 6; \\
|A_2 \cap A_4| &= \binom{3}{3} = 1.
\end{aligned}
$$

All other terms in the inclusion-exclusion expansions are 0. Thus finally

$$|A_0|-|A_1\cup A_2\cup A_3\cup A_4|=\binom{19}{3}-\binom{13}{3}-\binom{12}{3}-\binom{11}{3}-\binom{10}{3}+\binom{6}{3}+\binom{5}{3}+\binom{4}{3}+\binom{4}{3}+\binom{3}{3}=217. \qquad (10)$$

Exercise 1. How many ways are there to select 10 numbers from 1, 2, ..., 100 such that no two selected numbers are consecutive?

Exercise 2. In how many ways can ten A's, six B's and five C's by lined up in a row so that no two B's are adjacent? (Answer:[1])

Exercise 3. Find the number of positive solutions of $x_1+x_2+x_3+x_4+x_5=25$ with the restriction that each x_i is odd.

2. Occupancy.

How many ways are there to put n balls into m boxes.

Balls different?	Boxes different?	Can boxes be empty?	Answer
Yes	Yes	Yes	m^n
Yes	Yes	No	$T(n,m)$
No	Yes	Yes	$\binom{n+m-1}{m-1}$
No	Yes	No	$\binom{n-1}{m-1}$
Yes	No	Yes	$S(n,1)+\cdots+S(n,m)$
Yes	No	No	$S(n,m)$
No	No	Yes	$p_m(n+m)$
No	No	No	$p_m(n)$

where

$$T(n,m)=m^n-\binom{m}{1}(m-1)^n+\binom{m}{2}(m-2)^n+\cdots+(-1)^{m-1}\binom{m}{m-1}1^n. \qquad (11)$$

and

$$S(n,m)=\frac{1}{m!}T(n,m). \qquad (12)$$

Example 4. How many ways are there to distribute 18 toys to six children if each child receives a toy and

a) The toys are identical?

b) The toys are all different?

Solution.

a) We need to find the number of integer solutions to

$$x_1+\cdots+x_6=18, \qquad x_i>0 \qquad (13)$$

which is $C(17,5)$.

b) The answer is

$$T(18,6)=6^{18}-6\times 5^{18}+15\times 4^{18}-20\times 3^{18}+15\times 2^{18}-6. \qquad (14)$$

Example 5. At the end of the day, a bakery has seven oatmeal cookies, eight sugar cookies, and nine chocolate chip cookies.

a) In how many ways can the cookies be distributed to two different employees so that each employee receives at least one cookie?

b) In how many ways in part (a) will both employees receive 12 cookies?

1. $C(15,5)\,C(16,6)$.

Solution.

a) We note that there are exactly two ways to distribute the cookies that one of the employees has no cookie. So now we will simply ignore the "at least one cookie" requirement and just subtract our answer by 2 at the end.

We carry out the distribution in three steps.

1. Distribute 7 oatmeal cookies. This can be modelled as

$$x_1 + x_2 = 7, \qquad x_1, x_2 \geqslant 0. \tag{15}$$

   There are 8 different ways.

2. Distribute 8 sugar cookies. There are 9 ways.

3. Distribute 9 chocolate chip cookies. There are 10 ways.

Clearly the three steps are indepencent. Therefore the answer is $8 \times 9 \times 10 - 2 = 718$.

b) The problem is equivalent to "how many ways are there to take 12 cookies from 7 oatmeal cookies, 8 sugar cookies, and 9 chocolate chip cookies?"

We see that this in turn is equivalent to the following problem: How many different integer solutions are there for

$$x_1 + x_2 + x_3 = 12, \qquad 0 \leqslant x_1 \leqslant 7, \ 0 \leqslant x_2 \leqslant 8, \ 0 \leqslant x_3 \leqslant 9. \tag{16}$$

We solve this as follows.

- N_0: $x_1 + x_2 + x_3 = 12, x_1, x_2, x_3 \geqslant 0$. $N_0 = \binom{14}{2} = 91$.
- N_1: $x_1 + x_2 + x_3 = 12, x_1 \geqslant 8, x_2, x_3 \geqslant 0$. $N_1 = \binom{6}{2} = 15$.
- N_2: $x_1 + x_2 + x_3 = 12, x_1 \geqslant 0, x_2 \geqslant 9, x_3 \geqslant 0$. $N_2 = \binom{5}{2} = 10$.
- N_3: $x_1 + x_2 + x_3 = 12, x_1, x_2 \geqslant 0, x_3 \geqslant 10$. $N_3 = \binom{4}{2} = 6$.
- N_4: $x_1 + x_2 + x_3 = 12, x_1 \geqslant 8, x_2 \geqslant 9, x_3 \geqslant 0$. $N_4 = 0$.
- N_5: $x_1 + x_2 + x_3 = 12, \ x_1 \geqslant 8, x_2 \geqslant 0, x_3 \geqslant 10$. $N_5 = 0$.
- N_6: $x_1 + x_2 + x_3 = 12, \ x_1 \geqslant 0, \ x_2 \geqslant 9, x_3 \geqslant 10$. $N_6 = 0$.
- N_7: $x_1 + x_2 + x_3 = 12, \ x_1 \geqslant 8, x_2 \geqslant 9, x_3 \geqslant 10$. $N_7 = 0$.

Thus the answer is

$$91 - 15 - 10 - 6 + 0 + 0 + 0 - 0 = 60. \tag{17}$$

Exercise 4. How many ways are there to distribute 18 toys to six children if each child receives a toy and the 18 toys can be divided into three groups of 6,7,5 each, and the toys within each group are identical.

Exercise 5.

a) In how many ways can 25 identical books be assigned to five different book shelves?

b) In how many ways can 25 different books be assigned to five different bookshelves if the order of the books on each shelf is considered important?

Exercise 6. Show that the number of ways to partition a set of n distinct elements into k nonempty ordered lists equals $n! \, C(n-1, k-1)$.

3. Coloring.

It suffices to review the lecture on Jan. 27.