Identical balls, identical boxes.

We consider the following problem.

How many ways are there to distribute n identical balls into m identical boxes.

The boxes are not allowed to be empty

- We denote by $p_m(n)$ the number of ways to distribute n identical balls into m identical boxes with all boxes non-empty. It is clear that

$$p_m(n) = \text{number of ways to partition } n \text{ identical balls into } m \text{ groups.} \tag{1}$$

- Interpretations of $p_m(n)$.
  - We further notice that $p_m(n)$ is the number of integer solutions to

$$x_1 + \cdots + x_m = n, \qquad x_1 \geqslant x_2 \geqslant \cdots \geqslant x_m \geqslant 1. \tag{2}$$

    If we set $y_i = x_i - 1$, (2) becomes

$$y_1 + \cdots + y_m = n - m, \qquad y_1 \geqslant y_2 \geqslant \cdots \geqslant y_m \geqslant 0. \tag{3}$$

    This gives

$$p_m(n) = \sum_{s=1}^{m} p_s(n-m). \tag{4}$$

    Exercise 1. Prove (4).

  - Another way of understanding $p_m(n)$ is through the so-called Ferrer's graph.

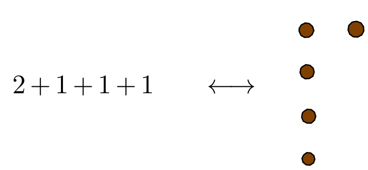

Figure 1. Ferrer's Graph

Using Ferrer's graph we can easily prove the following:

Proposition 1. $p_m(n) =$ *the number of partitions of* n *into summands whose largest is* m.

Proof. We prove the following: The number of partitions into exactly m summands is the same as the number of partitions with largest summand m. To see this we just notice that there is a bijection between the two sets given by

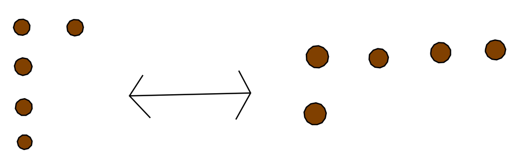

Figure 2. $2+1+1+1 \longleftrightarrow 4+1$

Thus ends the proof. □

Exercise 2. Use Ferrer's graph to prove the following:

the number of partitions of $n+m$ into m parts = the number of partitions of n into no more than m parts. (5)

Exercise 3. Show that the number of partitions of an integer n into summands of even size is equal to the number of partitions into summands such that each summand occurs an even number of times.

Exercise 4. A partition is said to be self-conjugate if the Ferrer's graph of the partition is equal to its own transpose. Show that the number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts.

- More properties of $p_m(n)$.
  - Clearly $p_1(n) = 1$.
  - It is also easy to see that $p_2(n) = \left\lfloor \frac{n}{2} \right\rfloor$, the largest integer no more than $n/2$.
  - We prove the following non-trivial result:

$$p_3(n) = \left\{ \frac{n^2}{12} \right\} \tag{6}$$

where $\{x\}$ denotes the integer nearest to $\frac{n^2}{12}$.

Exercise 5. Show that it is not possible to have $\frac{n^2}{12} = l + \frac{1}{2}$ for some integer l. Thus $\left\{ \frac{n^2}{12} \right\}$ is always well-defined.

Proof. Let $a_3(n)$ denote the number of solutions of $n = x_1 + x_2 + x_3$, $x_1 \geqslant x_2 \geqslant x_3 \geqslant 0$. Then we have $a_3(n) = p_3(n+3)$. On ther other hand, writing $y_3 = x_3$, $y_2 = x_2 - x_3$, $y_1 = x_1 - x_2$, we see that $a_3(n)$ is the number of solutions of

$$n = y_1 + 2\,y_2 + 3\,y_3, \qquad y_i \geqslant 0. \tag{7}$$

Therefore we have

$$\sum_{n=0}^{\infty} a_3(n)\,x^n = (1-x)^{-1}\,(1-x^2)^{-1}\,(1-x^3)^{-1}. \tag{8}$$

Application of the method of partial fractions, we reach

$$\begin{aligned} \sum_{n=0}^{\infty} a_3(n)\,x^n &= \frac{1}{6\,(1-x)^3} + \frac{1}{4\,(1-x)^2} + \frac{17}{72\,(1-x)} \\ &\quad + \frac{1}{8\,(1+x)} + \frac{1}{9\,(1-\omega\,x)} + \frac{1}{9\,(1-\omega^2\,x)} \end{aligned} \tag{9}$$

where $\omega = e^{2\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}\,i$.

Recall that

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots. \tag{10}$$

Differentiting this we obtain

$$\frac{1}{(1-x)^2} = 1 + 2\,x + 3\,x^2 + \cdots + (n+1)\,x^n + \cdots \tag{11}$$

and

$$\frac{1}{(1-x)^3} = 2 + 3 \cdot 2 \cdot x + 4 \cdot 3 \cdot x^2 + \cdots + (n+2)\,(n+1)\,x^n + \cdots \tag{12}$$

Therefore

$$a_3(n) - \frac{1}{12}\,(n+3)^2 = -\frac{7}{72} + \frac{(-1)^n}{8} + \frac{\omega^n + \omega^{2n}}{9}. \tag{13}$$

This leads to

$$\left| a_3(n) - \frac{(n+3)^2}{12} \right| \leqslant \frac{7}{72} + \frac{1}{8} + \frac{2}{9} < \frac{1}{2} \tag{14}$$

and the conclusion follows. □