Different balls, identical boxes.

The boxes are not allowed to be empty.

- The problem is equivalent to dividing $\{1,2,...,n\}$ into m subsets.
- We denote the answer by $S(n,m)$, known as the Stirling number of the second kind.
- For example, when $n=4$, $m=2$ there holds $S(4,2)=7$:
  - First notice that there are only two situations if the balls are also identical: One set three, the other one; or one set one, the other three.
  - Thus we can list all seven different ways as

$$\{1,2\},\{3,4\};\quad \{1,3\},\{2,4\};\quad \{1,4\},\{2,3\}; \tag{1}$$

$$\{1,2,3\},\{4\};\quad \{1,2,4\},\{3\};\quad \{1,3,4\},\{2\};\quad \{2,3,4\},\{1\}. \tag{2}$$

- To solve the general situation, we notice that if we start to label the boxes, then each division in can be turned into $m!$ different distrbutions. Consequently we have

$$T(n,m)=m!\cdot S(n,m). \tag{3}$$

- $S(n,m)$ enjoy similar identities as $C(n,k)$.
  - $S(n,1)=1=S(n,n)$. The proof is trivial.
  - $S(n,m)=S(n-1,m-1)+m\,S(n-1,m)$.
    We give a combinatorial proof here. We note that n balls into m identical boxes can be done in two mutually exclusive ways.
    1. Ball 1 has its own box and the remaining $n-1$ balls are distributed into the remaining $m-1$ boxes, leaving no box empty. There are $S(n-1,m-1)$ ways doing this.
    2. Ball 1 does not have its own box. This could be achieved through putting balls 2,3,...,n into the m boxes, leaving no box empty, and then choose one of the boxes to put ball 1 in. Note that after balls 2,3,...,n have been put in, the boxes are not identical anymore. Thus ball 1 has m choices. There are $m\,S(n-1,m)$ ways of doing this.

    So overall there are $S(n-1,m-1)+m\,S(n-1,m)$ different ways.

    Exercise 1. Prove $S(n,m)=S(n-1,m-1)+m\,S(n-1,m)$ using (3) and the formula for $T(n,m)$.
  - $S(n,n-1)=C(n,2)$.
    We notice that putting n balls into $n-1$ boxes with no empty box must result in one box with two balls and all other $n-2$ boxes with one ball each.
    We do this in three steps.
    1. Choose two balls from the n different balls. There are $C(n,2)$ different ways to do this.
    2. Put these two balls into one box. There is one way to do this as all the boxes are identical.
    3. Put the remaining $n-2$ balls into the remaining $n-2$ boxes. As the boxes are identical, there is only one possible outcome, that is each ball has its own box.

    By the product rule we see that $S(n,n-1)=C(n,2)\times 1\times 1$ and the conclusion follows.

    Exercise 2. Give a similar proof for $S(n,n-2)=C(n,3)+3\,C(n,4)$.

Example 1. In how many ways can $n\geqslant 2$ travelers share two identical cabs with no cab empty?

Solution. We see that the answer is

$$\begin{aligned} S(n,2) &= S(n-1,1)+2\,S(n-1,2) \\ &= 1+2\,[S(n-2,1)+2\,S(n-2,2)] \\ &= 1+2\,[1+2\,[S(n-3,1)+2\,S(n-3,2)]] \\ &\vdots \\ &= 2^{n-1}-1. \end{aligned} \tag{4}$$

Remark 2. Alternatively, we have

$$\begin{aligned} S(n,2) &= \frac{1}{2!}T(n,2) \\ &= \frac{1}{2!}[2^n-2\cdot 1^n]=2^{n-1}-1. \end{aligned} \tag{5}$$

Example 3. In how many ways can we factor the integer 30,030 into three positive integers if the order of the factors is unimportant and each factor is greater than 1?

Solution. We have

$$30,030=2\times 3\times 5\times 7\times 11\times 13. \tag{6}$$

Thus the number of factorizations is

$$\begin{aligned} S(6,3) &= S(5,2)+3\,S(5,3) \\ &= S(4,1)+2\,S(4,2)+3\,[S(4,2)+3\,S(4,3)] \\ &= 1+2\,(2^3-1)+3\,[2^3-1+3\,[S(3,2)+3\,S(3,3)]] \\ &= 1+14+3\,[7+3\cdot(3+3)] \\ &= 1+14+75=90. \end{aligned} \tag{7}$$

Remark 4. Alternatively, we could use

$$\begin{aligned} S(6,3) &= \frac{1}{3!}T(6,3) \\ &= \frac{1}{6}[3^6-3\times 2^6+3\times 1^6] \\ &= \frac{1}{6}[729-192+3]=90. \end{aligned} \tag{8}$$

Exercise 3. Show by a combinatorial argument that

$$S(n+1,m)=C(n,m-1)\,S(m-1,m-1)+\cdots+C(n,n)\,S(n,m-1). \tag{9}$$

The boxes are allowed to be empty.

If we allow cells to be empty, then clearly the answer is

$$S(n,1)+\cdots+S(n,m). \tag{10}$$

We define the Bell number B_n as the number of partitions of a set of n different elements into nonempty, indistinguishable cells. Thus

$$B_n=S(n,0)+S(n,1)+\cdots+S(n,n). \tag{11}$$

Exercise 4. ([?]) Show that

$$B_n=\binom{n-1}{0}B_0+\binom{n-1}{1}B_1+\cdots+\binom{n-1}{n-1}B_{n-1}. \tag{12}$$