The problem:

> How many different ways are there to put n balls into m boxes?

There are three assumptions, leading to eight sub-problems:

1. The balls are different or identical;
2. The cells are different or identical;
3. The cells are allowed to be empty or not.

Different balls, different boxes

The boxes are allowed to be empty.

In this case we can order the balls by $1, 2, ...$ and the cells by $1, 2, ...$ Take ball 1. We can put it in cell 1 or 2 or 3 ... or m. In other words for each ball there are m choices of where to put it. Consequently the total number of ways are

$$m \times m \times m \times \cdots \times m = m^n. \tag{1}$$

Exercise 1. Critique the following argument.

> "For each box there are n choices of balls. Thus the total is n^m".

Exercise 2. How many 5-digit binary strings are there?

Exercise 3. How many 5-digit binary numbers are there?

The boxes are not allowed to be empty.

There are three cases.

- Case 1. $n < m$.
  In this case it is clear that the answer is 0.
- Case 2. $n = m$.
  In this case we can line up the boxes B1, B2, B3, ..., Bn. We observe the following.
  - Each way of putting the n balls into these boxes corresponds to a unique permutation of the numbers $1, 2, ..., n$. For example, when $n = 5$, if we put ball 1 into box 4, ball 2 into box 3, ball 3 into box 5, ball 4 into box 1, and ball 5 into box 2, after we line up the boxes as B1, B2, B3, B4, B5, the order of the balls are 45213, which is a permutation of 1, 2, 3, 4, 5.
  - On the other hand, every permutation corresponds to a unique way of putting these balls into boxes. For example, when $n = 5$, the permutation 54123 corresponds to putting ball 5 into box 1, ball 4 into box 2, ball 1 into box 3, ball 2 into box 4, ball 3 into box 5.

  Thus we see that there is a one-to-one correspondence between all the different ways of putting n different balls into n different boxes, and all the different permutations of $1, 2, ..., n$. Consequently the answer for this case is $n!$.
- Case 3. $n > m$.

  Exercise 4. Critique the following argument.

  > The problem is equivalent to first listing $1, 2, ..., n$ in any order, then putting $m - 1$ "separators" into the $n - 1$ spaces between the numbers, so that the balls with numbers before the first separator goes into the first box, the balls with numbers between the first and the second separator goes into the second box, and so on. Thus the answer should be
  >
  > $$n! \binom{n-1}{m-1}. \tag{2}$$

  - Denote the answer by $T(n, m)$.
  - Consider for example $m = 3$.

If we ignore the requirement that the boxes cannot be empty, there are 3^n different ways to put the n balls into the 3 boxes.

However this is an over-count and we have to take out those "illegal" ways, which are characterized by "at least one box is empty".

Now if we just add the extra requirement that box 1 is empty, we obtain 2^n ways. Same for boxes 2 and 3. Thus we should subtract 3×2^n.

However $3^n - 3 \times 2^n$ is an under-count. For example, consider putting all the n balls into box 3. This is counted once in the 2^n when requiring box 1 to be empty, and is counted again in the 2^n when requiring box 2 to be empty. We should add them back. There are three cases: boxes 1&2 are empty, 2&3 are empty, and 3&1 are empty. Thus overall we add back 3×1^n.

Finally the answer is $3^n - 3 \times 2^n + 3 \times 1^n$.

Exercise 5. Compare with the "inclusion-exclusion principle" we discussed before. What happens to the case "all three boxes are empty"? Why isn't the answer $3^n - 3 \times 2^n + 3 \times 1^n - 1$?

- The inclusion-exclusion principle.

THEOREM 1. (THE INCLUSION-EXCLUSION PRINCIPLE) *If $A_1, \ldots, A_n$ are n arbitrary sets, then*

$$\begin{aligned}|A_1 \cup \cdots \cup A_n| &= \sum_{i=1}^{n} |A_i| \\ &- \sum_{i,j=1, i\neq j}^{n} |A_i \cap A_j| \\ &+ \sum_{i,j,k=1, i,j,k \text{ distinct}}^{n} |A_i \cap A_j \cap A_k| \\ &- \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|. \end{aligned} \tag{3}$$

- General situation.

  Application of the inclusion-exclusion principle now gives

$$T(n,m) = m^n - \binom{m}{1}(m-1)^n + \binom{m}{2}(m-2)^n - \cdots + (-1)^{m-1}\binom{m}{m-1} 1^n. \tag{4}$$

Exercise 6. How many ways are there to put 5 different balls into 4 different boxes?

IDENTICAL BALLS, DIFFERENT BOXES.

The boxes are allowed to be empty.

We see that this is the same as the problem of counting integer solutions to

$$x_1 + \cdots + x_m = n, \qquad x_i \geqslant 0. \tag{5}$$

Therefore the answer is

$$\binom{n+m-1}{m-1}. \tag{6}$$

The boxes are allowed to be empty.

We see that this is the same ase the problem of counting integer solutions to

$$x_1 + \cdots + x_m = n, \qquad x_i > 0. \tag{7}$$

Therefore the answer is

$$\binom{n-1}{m-1}. \tag{8}$$