Review

Permutations and Combinations.

- How many different ways are there to list n distinct objects in a line?

$$P(n) := n! = n \cdot (n-1) \cdot \cdots \cdot 1. \tag{1}$$

- How many different ways are there to list m objects from n distinct objects into a line?

$$P(n,m) := \frac{n!}{(n-m)!}. \tag{2}$$

- How many different ways are there to list n objects which consists of k groups of $n_1, \ldots, n_k$ identical objects respectively?

$$\binom{n}{n_1, \ldots, n_k} := \frac{n!}{n_1! \cdots n_k!}. \tag{3}$$

  - $k=2$, $n_1 = m$, $n_2 = n - m$. Combination numbers

$$C(n,m) = \binom{n}{m} = \frac{n!}{m!\,(n-m)!}. \tag{4}$$

- Alternative interpretations.
  - $C(n,m)$: How many different ways to pick m objects from n distinct objects?
  - $\binom{n}{n_1, \ldots, n_k}$: How many different ways to divide n distinct objects into k groups of $n_1, \ldots, n_k$ objects each?

QUESTION 1. *Is there a simple formula for the arranging of k objects from a collection of n objects consisting of m groups of $n_1, \ldots, n_m$ identical objects?*

Example 2. At the end of the 2nd Harry Potter movie Tom Riddle wrote "TOM MARVOLO RIDDLE" in the air and manipulated them into "I AM LORD VOLDEMORT". Suppose he claimed that this is an accident. Calculate the probability that this happened by chance by calculating the total number of ways of re-arranging the letters (ignore the spaces).

Solution. Listing the letters as A DD E I LL MM OOO RR T V. Total is 16 letters and 10 groups. The total number of possible re-arrangements is

$$\binom{16}{1,2,1,1,2,2,3,2,1,1} = \frac{16!}{2!\,2!\,2!\,2!\,3!} = 217945728000. \tag{5}$$

So the probability of this happening by chance is

$$\frac{1}{217945728000} \approx 4.59 \times 10^{-12}. \tag{6}$$

Exercise 1. How many permutations are there of the letters, taken all at a time, of the words

a) assesses,

b) humuhumunukunukuapuaa (Hawaiian word for a species of fish).

Exercise 2. Consider a three-dimensional steel framework; how many different paths of length fifteen units are there from one intersection point in the framework to another that is located four units tot he right, five unites back, and six units up?

Exercise 3. In how many different orders can the following 17 letters be written?

$$xxxxyyyyyzzzzzzww \tag{7}$$

Exercise 4. Show that

$$(x+y+z+w)^n = \sum_{n_1+\cdots+n_4=n,\, n_i \geqslant 0} \binom{n}{n_1, n_2, n_3, n_4} x^{n_1} y^{n_2} z^{n_3} w^{n_4}. \tag{8}$$

Exercise 5. Show that

$$\sum_{n_1+n_2+n_3=n,\, n_i \geqslant 0} (-1)^{n_3} \binom{n}{n_1, n_2, n_3} = 1. \tag{9}$$

Integer Solutions

We consider the following problem:

> How many integer solutions are there for the equation
> $$x_1 + \cdots + x_m = n. \tag{10}$$

- It is clear that if there is no restriction for each $x_1, \ldots, x_m$, there are infinitely many solutions.
- We will consider the following restrictions:
  - $x_i > 0$, $i = 1, 2, \ldots, m$;
  - $x_i \geqslant 0$, $i = 1, 2, \ldots, m$;
  - $a_i \leqslant (<) x_i \leqslant (<) b_i$.

$$\boldsymbol{x_1 + \cdots + x_m = n.}$$

Positive solutions.

- To solve this we consider the following modeling:
  $$1 + 1 + \cdots + 1 \tag{11}$$
  where there are n 1's. To obtain one solution of $x_1 + \cdots + x_m = n$, we replace m of the "+" symbols by ",", and carry out the remaining summations.
- Therefore the number of positive solutions is $\cancel{C(n-1, m)}$. $C(n-1, m-1)$

Non-negative solutions.

- Let $y_i = x_i + 1$. Then $y_i > 0$ and
  $$y_1 + \cdots + y_m = n + m. \tag{12}$$
  - On one hand, it is clear that every non-negative solution to $x_1 + \cdots + x_m = n$ gives a positive solution to (12). Furthermore different $(x_1, \ldots, x_m)$ leads to different $(y_1, \ldots, y_m)$.
  - On the other hand, given a positive solution to (12), we can obtain a non-negative solution to $x_1 + \cdots + x_m = n$ through setting $x_i := y_i - 1$.
- Therefore the number of non-negative solutions to $x_1 + \cdots + x_m = n$ is given by $\cancel{C(n+m-1, m)}$.

$C(n+m-1, m-1)$