Math 421 Q1 Winter 2017 Homework 8 Solutions

Due Mar. 30, 12pm.

Total 20 points

Question 1. (5 pts) *Prove that in any simple graph, there are two vertices having the same degrees.*

Proof. Let the order of the graph be n. Then the maximum degree of a vertex is $n-1$. For no two vertices to have the same degree we must have the degree sequence $(n-1, n-2, ..., 0)$. However this is not possible as if there is a vertex with degree $n-1$, it must be connected to every other vertex and consequently no vertex can have degree 0. $\square$

Question 2. (5 pts) *Let $G=(V,E)$ with $V=10$ and $E=26$. Prove that there is a vertex in G whose degree is no less than 6.*

Proof. Let the degree sequence of G be $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{10} \geqslant 0$. We now have

$$52 = 2\,|E| = d_1 + \cdots + d_{10} \leqslant 10\,d_1. \tag{1}$$

Therefore $d_1 \geqslant 6$. $\square$

Question 3. (5 pts) *A sequence $(d_1, ..., d_n)$ is called "graphic" if and only if it is the degree sequence of some simple graph. Prove that the sequence $(6, 6, 5, 4, 2, 2, 1)$ is not graphic.*

Proof. Assume that it is the degree sequence of a simple graph G. Then G has seven vertices $v_1, ..., v_7$. We mark $v_1, ..., v_7$ in such a way that $\deg(v_1) = 6, \deg(v_2) = 6, \deg(v_3) = 5, ..., \deg(v_7) = 1$.

As $\deg(v_1) = 6 = 7 - 1$, v_1 is connected to every other vertex. Now let $G' = (\{v_2, ..., v_7\}, E')$ be the simple graph obtained from G by deleting v_1 and the six edges from v_1. The degree sequence of G' is then $(5, 4, 3, 1, 1, 0)$. However this leads to contradiction: As $\deg(v_2) = 5$, all five possible edges $\{v_2, v_3\}, \{v_2, v_4\}, ...,$ $\{v_2, v_7\}$ are present in G'. In particular we have $\deg(v_7) \geqslant 1$. Contradiction. $\square$

Question 4. (5 pts) *Prove that in any group of 10 people there are either four mutual friends or three mutual strangers (non-friends, see last example of Mar. 22 lecture notes if you don't understand the problem).*

Proof. Represent the 10 people by 10 vertices $v_1, ..., v_{10}$ in a simple graph G and connect each pair of friends by an edge. We discuss the following cases.

- $\deg(v_1) \leqslant 5$. In this case without loss of generality we can assume that there are no edges connecting v_1 to v_2, v_3, v_4, v_5. Now consider the six pairs $\{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}$. If any one pair is not connected, we would have three mutual strangers (for example if there is no edge between v_2, v_3, then v_1, v_2, v_3 are three mutual strangers). On the other hand, if all six pairs are connected by edges, then v_2, v_3, v_4, v_5 are four mutual friends.
- $\deg(v_1) \geqslant 6$. In this case without loss of generality assume v_1 is connected to $v_2, ..., v_7$. Now we know that among $v_2, ..., v_7$ there are either three mutual friends or three mutual strangers. The proof ends by noticing that in the former case, these three together with v_1 form a group of four mutual friends. $\square$

Remark 5. It is clear that the same method proves the existence of either four mutual strangers or three mutual friends. Does this, together with the claim in Question 4, mean that there are either four mutual friends or four mutual strangers?