Math 421 Winter 2017 Midterm 2 Solutions

Mar. 17, 2017 1pm - 1:50pm. Total 30 Pts

NAME: ID#:

- Please write clearly and **show enough work/explain your reasoning** (this is very important!).
- No electronic devices are allowed.

Question 1. (8 pts) *Let $a_{n+3} = 3\, a_{n+2} - 3\, a_{n+1} + a_n$ for all $n \geqslant 0$ and $a_0 = a_1 = 0, a_2 = 1$. Use generating function to find the numerical value of a_{100}.*

Solution. We have

$$\begin{aligned} A(x) &:= \sum_{n=0}^{\infty} a_n\, x^n \\ &= x^2 + \sum_{n=3}^{\infty} a_n\, x^n \\ &= x^2 + \sum_{n=3}^{\infty} (3\, a_{n-1} - 3\, a_{n-2} + a_{n-3})\, x^n \\ &= x^2 + 3 \sum_{n=2}^{\infty} a_n\, x^{n+1} - 3 \sum_{n=1}^{\infty} a_n\, x^{n+2} + \sum_{n=0}^{\infty} a_n\, x^{n+3} \\ &= x^2 + 3\, x\, A(x) - 3\, x^2\, A(x) + x^3\, A(x). \end{aligned} \tag{1}$$

This gives

$$\begin{aligned} A(x) &= \frac{x^2}{(1-x)^3} \\ &= x^2 \sum_{n=0}^{\infty} \frac{(n+2)\,(n+1)}{2}\, x^n \\ &= \sum_{n=2}^{\infty} \frac{n\,(n-1)}{2}\, x^n. \end{aligned} \tag{2}$$

Thus $a_{100} = \frac{100 \times 99}{2} = 4950$.

Question 2. **(7 pts)** *Find the number of different ways distributing* n *different balls to four boxes where an odd number of balls are in the fourth box.*

Solution. The generating function is

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n =: E(x) &= \left(1+x+\frac{x^2}{2!}+\cdots\right)^3 \left(x+\frac{x^3}{3!}+\cdots\right) \\ &= e^{3x}\left(\frac{e^x-e^{-x}}{2}\right) \\ &= \frac{1}{2}(e^{4x}-e^{2x}) \\ &= \sum_{n=0}^{\infty} \frac{4^n-2^n}{2\cdot n!} x^n. \end{aligned} \tag{3}$$

Therefore $a_n = \frac{4^n-2^n}{2}$.

Question 3. **(12 pts)** *Find the number of ways to color the eight vertices of a regular octagon with 2 colors, if*

a) **(6 pts)** *only rotations are allowed;*

b) **(6 pts)** *both rotation and flipping are allowed.*

Solution. Mark the eight vertices, from the top left, counter-clockwise, 1,2,...,8.

a) Then the eight rotations are:

- $i=(1)(2)\cdots(8)$, $c(i)=8$;
- $g_1=(12345678)$, $c(g_1)=1$;
- $g_2=(1357)(2468)$, $c(g_2)=2$;
- $g_3=(14725836)$, $c(g_3)=1$;
- $g_4=(15)(26)(37)(48)$, $c(g_4)=4$;
- $g_5=(16385274)$, $c(g_5)=1$;
- $g_6=(1753)(2864)$, $c(g_6)=2$;
- $g_7=(18765432)$, $c(g_7)=1$.

Thus the answer is

$$\frac{2^8+4\times 2^1+2\times 2^2+2^4}{8}=36. \tag{4}$$

b) Besides $i, g_1, \ldots, g_7$, we now have eight flippings:

- $f_1 = (18)(27)(36)(45)$, and three others also having $c(g) = 4$;
- $f_5 = (1)(5)(28)(37)(46)$, and three others also having $c(g) = 5$.

Therefore the answer is

$$\frac{2^8 + 4 \times 2^1 + 2 \times 2^2 + 2^4 + 4 \times 2^4 + 4 \times 2^5}{16} = 30. \tag{5}$$

Question 4. (3 pts) *How many ways are there to put 16 identical balls in four identical boxes at the four vertices (one at each vertex) of a square board, allowing empty boxes, assuming that the board can freely rotate? Give your answer in numerical value.*

Solution. We mark the four boxes $1, 2, 3, 4$, counter-clockwise. We set X to be the collection of all possible ways putting 24 identical balls into four different boxes. Thus we have $|X| = \binom{16+3}{3}$. Now the symmetry group consists of identity and three rotations, $G = \{i, r_1, r_2, r_3\}$. We calculate

- $X_i = X$ so $|X_i| = \binom{19}{3} = 969$.
- X_{r_1}. We see that all four boxes must have the same number of balls, so $|X_{r_1}| = 1$.
- X_{r_2}. Boxes 1 and 3 must have the same number of balls, and boxes 2 and 4 must have the same number of balls. So $|X_{r_2}|$ equals the number of integer solutions to $x_1 + x_2 = 8, x_i \geqslant 0$. Therefore $|X_{r_2}| = 9$.
- $|X_{r_3}| = 1$ similar to X_{r_1}.

Thus the answer is

$$\frac{969 + 1 + 9 + 1}{4} = 245. \tag{6}$$