Math 421 Q1 Winter 2017 Homework 6 Solutions

Due Mar. 9, 12pm.

Total 20 points

Question 1. (10 pts) *How many ways are there to color the corners of a regular pentagon with the colors red, blue, and green if*

a) (5 pts) *Only rotation is allowed.*

b) (5 pts) *Both rotation and flipping are allowed.*

Give the numerical value of your answer.

Solution.

a) The rotation group G of a regular pentagon has 5 elements:

$$g_0 = i, \qquad g_1 = \text{clockwise rotation of } \frac{2\,\pi}{5}, \quad g_2 = g_1^2, \quad g_3 = g_1^3, \quad g_4 = g_1^4. \tag{1}$$

The set X consists of all possible colorings of the five (marked) corners with three colors, thus $|X| = 3^5 = 243$.

We calculate

- $X_i = X$. So $|X_i| = 243$.
- X_{g_1}. If a coloring does not change under g_1, then necessarily all five corners are of the same color. So $|X_{g_1}| = 3$.
- By similar argument, $|X_{g_2}| = |X_{g_3}| = |X_{g_4}| = 3$.

Therefore the answer is

$$\frac{243 + 3 + 3 + 3 + 3}{5} = 51. \tag{2}$$

b) This time the group G has ten elements:

$$g_0 = i, \qquad g_1 = \text{clockwise rotation of } \frac{2\,\pi}{5}, \quad g_2 = g_1^2, \quad g_3 = g_1^3, \quad g_4 = g_1^4, \tag{3}$$

together with $f_1, ..., f_5$, the five "flip"s around the line passing one corner and the middle point of the opposite side.

- We still have $|X_{g_0}| = 243$, $|X_{g_i}| = 3$ for $i = 1, 2, 3, 4$.
- We see that for a coloring to be "fixed" by f_1, we can only freely color the "left" side. Therefore $|X_{f_1}| = 3 \times 3 \times 3 = 27$.
- Clearly $|X_{f_i}| = |X_{f_1}| = 27$, $i = 2, 3, 4, 5$.

Consequently we have by Burnside's Lemma,

$$\text{Ans} = \frac{243 + 3 + 3 + 3 + 3 + 27 + 27 + 27 + 27 + 27}{10} = 39. \tag{4}$$

Question 2. (5 pts) *How many ways are there to color the vertices of a square with five colors such that no color is used on more than three vertices?*

Solution. Note that if one color is used on more than three vertices, then all four vertices of the square are colored by this same color. Thus there are exactly 5 ways to color the vertices so that the requirement "no color is used on more than three vertices" is not satisfied.

Take X to be all possible colorings of four marked vertices with five colors. We have $|X| = 5^4 = 625$.

For the group G, we see that there are 8 elements:

- Four rotations of $0, \pi/2, \pi, 3\pi/2$. We denote them r_0, r_1, r_2, r_3.
- Two flippings around the diagonals. We denote them d_1, d_2.
- Two flippings around the line passing middle points of opposite sides. We denote them f_1, f_2.

Now we calculate

- $|X_{r_0}| = 5^4 = 625$;
- $|X_{r_1}| = 5$.
- $|X_{r_2}| = 5^2 = 25$. Note that the rotation r_2 switches diagonal vertices. Therefore the two pairs of diagonal vertices can be colored differently.
- $|X_{r_3}| = 5$.
- $|X_{d_1}| = |X_{d_2}| = 5^3 = 125$.
- $|X_{f_1}| = |X_{f_2}| = 25$.

Now by Burnside's Lemma, there are

$$\frac{625 + 5 + 25 + 5 + 125 + 125 + 25 + 25}{8} = 120 \tag{5}$$

different ways to color the four vertices with five colors. Subtracting the five colorings that does not satisfy the extra requirement, we arrive at 115 as the answer.

QUESTION 3. (5 PTS) *How many ways are there to color the four vertices of a non-square rhombus with m colors? Both rotation and flipping are allowed.*

Solution. We see that $|X| = m^4$.

For the symmetry group of the rhombus, we see that it consists of

- i: identity;
- r: rotation by π around the center.
- f_l, f_s: flippings around the long and short diagonals.

We calculate

- $|X_i| = m^4$;
- $|X_r| = m^2$;
- $|X_{f_l}| = |X_{f_s}| = m^3$.

Therefore the answer is

$$\frac{m^4 + 2m^3 + m^2}{4} = \frac{(m+1)^2 m^2}{4}. \tag{6}$$