Math 421 Q1 Winter 2017 Homework 4 Solutions

Due Feb. 16, 12pm.

Total 20 points.

Question 1. (5 pts) *Use generating function to count the number of selections of 30 toys from 10 different types of toys if at least two but no more than five of each kind must be selected.* ***You can leave the answer as combination numbers and do not need to calculate the numerical value.***

Solution. The generating function is

$$\begin{aligned}(x^2+x^3+x^4+x^5)^{10} &= x^{20}(1+x+x^2+x^3)^{10} \\ &= x^{20}\left[\frac{1-x^4}{1-x}\right]^{10} \\ &= x^{20}(1-x^4)^{10}\frac{1}{(1-x)^{10}} \\ &= \frac{x^{20}(1-x^4)^{10}}{9!}\sum_{n=0}^{\infty}(n+9)\cdots(n+1)\,x^n. \end{aligned} \tag{1}$$

As

$$(1-x^4)^{10} = 1-\binom{10}{1}x^4+\binom{10}{2}x^8+\text{terms with power at least 12,} \tag{2}$$

we see that the coefficient of x^{30} is given by

$$\frac{1}{9!}\left[19\times\cdots\times 11-\binom{10}{1}15\times\cdots\times 7+\binom{10}{2}11\times\cdots 3\right]=\binom{19}{9}-10\binom{15}{9}+45\binom{11}{9}. \tag{3}$$

Question 2. (5 pts) *Use generating function method to solve the following: In how many ways can 25 identical balls be distributed to nine different boxes if each box receives an odd number of balls?* ***Give your answer in numerical value.***

Solution. The generating function is

$$(x+x^3+x^5+\cdots)^9 = x^9(1+x^2+x^4+\cdots)^9 = x^9\left(\frac{1}{1-x^2}\right)^9. \tag{4}$$

The answer is the coefficient of x^{16} in the expansion of $\left(\frac{1}{1-x^2}\right)^9$, which is the same as the coefficient of z^8 in the expansion of $\left(\frac{1}{1-z}\right)^9 = \frac{1}{8!}\sum_{n=0}^{\infty}(n+8)\cdots(n+1)\,x^n$ which is $C(16,8)=12870$.

Question 3. (5 pts) *Use generating functions to find the number of integer solutions to*

$$x_1+x_2+x_3=16, \qquad x_i\geqslant 0, \quad x_1 \textit{ odd}, x_2 \textit{ even}, x_3 \textit{ prime}. \tag{5}$$

Give your answer in numerical value.

Solution. The generating function is

$$A(x) = (x+x^3+x^5+\cdots)(1+x^2+x^4+\cdots)\sum_{p \text{ prime}} x^p = \frac{x}{(1-x^2)^2}\sum_{p \text{ prime}} x^p. \tag{6}$$

Using

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)\,x^n \Longrightarrow \frac{1}{(1-x^2)^2}=\sum_{n=0}^{\infty}(n+1)\,x^{2n}, \tag{7}$$

we see that

$$\begin{aligned} A(x) &= \left(\sum_{p \text{ prime}} x^{p+1}\right)\left(\sum_{n=0}^{\infty}(n+1)\,x^{2n}\right) \\ &= x^3\sum_{n=0}^{\infty}(n+1)\,x^{2n} + x^4\sum_{n=0}^{\infty}(n+1)\,x^{2n} + x^6\sum_{n=0}^{\infty}(n+1)\,x^{2n} \\ &\quad + x^8\sum_{n=0}^{\infty}(n+1)\,x^{2n} + x^{12}\sum_{n=0}^{\infty}(n+1)\,x^{2n} + x^{14}\sum_{n=0}^{\infty}(n+1)\,x^{2n} \\ &\quad + \text{terms with power of } x \text{ more than 15.} \end{aligned} \tag{8}$$

Consequently the answer is given by

$$(6+1)+(5+1)+(4+1)+(2+1)+(1+1)=23. \tag{9}$$

QUESTION 4. (5 PTS) *Show that the number of partitions of n into summands not divisible by 3 is equal to the number of partitions of n where no summand occurs more than twice.*

Proof. The generating function for the former is

$$(1+x+x^2+\cdots)\,(1+x^2+x^4+\cdots)\,(1+x^4+x^8+\cdots)=\frac{1}{(1-x)\,(1-x^2)\,(1-x^4)\,(1-x^5)\cdots}. \tag{10}$$

The generating function for the latter is

$$(1+x+x^2)\,(1+x^2+x^4)\,(1+x^3+x^6)\cdots. \tag{11}$$

We have

$$\begin{aligned} (1+x+x^2)\,(1+x^2+x^4)\,(1+x^3+x^6)\cdots &= \frac{1-x^3}{1-x}\,\frac{1-x^6}{1-x^2}\,\frac{1-x^9}{1-x^3}\,\frac{1-x^{12}}{1-x^4}\cdots \\ &= \frac{1}{(1-x)\,(1-x^2)\,(1-x^4)\,(1-x^5)\cdots}. \end{aligned} \tag{12}$$

Thus ends the proof. $\square$