Midterm 2

(Nov. 9, 2017, 11:10am–12:10pm. Total 15+2 pts)

NAME: ID#:

- There are three regular problems and one bonus problem (total 15 pts + 2 bonus pts).
- Please write clearly and show enough work.
- You may find the following formulas useful.

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{E}\,\dot{u} + \mathbb{F}\,\dot{v}) = \frac{1}{2}\,(\mathbb{E}_u\,\dot{u}^2 + 2\,\mathbb{F}_u\,\dot{u}\,\dot{v} + \mathbb{G}_v\,\dot{v}^2), \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{F}\,\dot{u} + \mathbb{G}\,\dot{v}) = \frac{1}{2}\,(\mathbb{E}_v\,\dot{u}^2 + 2\,\mathbb{F}_v\,\dot{u}\,\dot{v} + \mathbb{G}_v\,\dot{v}^2). \tag{2}$$

$$\ddot{u} + \Gamma^1_{11}\,\dot{u}^2 + 2\,\Gamma^1_{12}\,\dot{u}\,\dot{v} + \Gamma^1_{22}\,\dot{v}^2 = 0, \tag{3}$$

$$\ddot{v} + \Gamma^2_{11}\,\dot{u}^2 + 2\,\Gamma^2_{12}\,\dot{u}\,\dot{v} + \Gamma^2_{22}\,\dot{v}^2 = 0. \tag{4}$$

QUESTION 1. (5 PTS) *Consider the surface S given by $\sigma(u,v) = (u, v, u^2 - v^2)$. Calculate $\cos\theta$ where θ is the angle between the curves $u = 2$ and $u = 2v$.*

Solution.

1. Parametrize the curves.
   - $u = 2$: $u(t) = 2, v(t) = t$;
   - $u = v$: $u(t) = 2t, v(t) = t$.
2. The intersection point is $u = 2, v = 1$.
3. Calculate $\mathbb{E}, \mathbb{F}, \mathbb{G}$ at $\sigma(2,1)$.

$$\sigma_u = (1, 0, 2u), \sigma_v = (0, 1, -2v) \Longrightarrow \sigma_u(2,1) = (1,0,4), \sigma_v(2,1) = (0,1,-2). \tag{5}$$

   Thus

$$\mathbb{E}(2,1) = 17, \qquad \mathbb{F}(2,1) = -8, \qquad \mathbb{G}(2,1) = 5. \tag{6}$$

4. Calculate the tangent vectors: $v = \sigma_v, w = 2\sigma_u + \sigma_v$ so $v_1 = 0, v_2 = 1, w_1 = 2, w_2 = 1$.
5. Calculate the angle.

$$\begin{aligned} \cos\theta &= \frac{\mathbb{E}\, v_1 w_1 + \mathbb{F}\,(v_1 w_2 + v_2 w_1) + \mathbb{G}\, v_2 w_2}{(\mathbb{E}\, v_1^2 + 2\,\mathbb{F}\, v_1 v_2 + \mathbb{G}\, v_2^2)^{1/2}\, (\mathbb{E}\, w_1^2 + 2\,\mathbb{F}\, w_1 w_2 + \mathbb{G}\, w_2^2)^{1/2}} \\ &= \frac{-8 \cdot 1 \cdot 2 + 5 \cdot 1 \cdot 1}{5^{1/2}\,(17 \cdot 2^2 - 16 \cdot 2 \cdot 1 + 5 \cdot 1^2)^{1/2}} \\ &= \frac{-11}{205^{1/2}}. \end{aligned} \tag{7}$$

Question 2. (5 pts) *Consider the surface S given by $\sigma(u,v)=(u,v,u^2-v^2)$. Calculate the principal, mean, and Gaussian curvatures at $p=(1,0,1)$.*

Solution. We have $p=\sigma(1,0)$. Furthermore

$$\sigma_u=(1,0,2u), \sigma_v=(0,1,-2v) \Longrightarrow \sigma_u(1,0)=(1,0,2), \sigma_v(1,0)=(0,1,0). \tag{8}$$

Thus

$$N(p)=\frac{(1,0,2)\times(0,1,0)}{\|(1,0,2)\times(0,1,0)\|}=\frac{(-2,0,1)}{\|(-2,0,1)\|}=\frac{1}{\sqrt{5}}(-2,0,1). \tag{9}$$

Furthermore we have

$$\sigma_{uu}=(0,0,2), \quad \sigma_{uv}=(0,0,0), \quad \sigma_{vv}=(0,0,-2). \tag{10}$$

Thus at p,

$$\mathbb{E}=5, \qquad \mathbb{F}=0, \qquad \mathbb{G}=1, \tag{11}$$

$$\mathbb{L}=\frac{2}{\sqrt{5}}, \qquad \mathbb{M}=0, \qquad \mathbb{N}=\frac{-2}{\sqrt{5}}. \tag{12}$$

Solving

$$\det\left[\begin{pmatrix} \frac{2}{\sqrt{5}} & 0 \\ 0 & -\frac{2}{\sqrt{5}} \end{pmatrix} - \kappa \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}\right]=0 \tag{13}$$

we have at p,

$$\kappa_1=\frac{2}{5\sqrt{5}}, \qquad \kappa_2=-\frac{2}{\sqrt{5}}. \tag{14}$$

Consequently

$$H=\frac{-4}{5\sqrt{5}}, \qquad K=\frac{-4}{25}. \tag{15}$$

QUESTION 3. (5 PTS) *Consider the surface S given by $\sigma(u,v)=(u,v,u^2-v^2)$.*

a) (3 PTS) *Calculate Γ_{ij}^k.*

b) (2 PTS) *Can $v=0$ be parametrized into a geodesic? Justify your claim.*

Solution.

a) We have

$$\sigma_u=(1,0,2\,u), \qquad \sigma_v=(0,1,-2\,v), \qquad N=\frac{(-2\,u,2\,v,1)}{\sqrt{1+4\,u^2+4\,v^2}}. \tag{16}$$

$$\sigma_{uu}=(0,0,2), \qquad \sigma_{uv}=(0,0,0), \qquad \sigma_{vv}=(0,0,-2). \tag{17}$$

Thus

$$\mathbb{L}=\frac{2}{\sqrt{1+4\,u^2+4\,v^2}}, \qquad \mathbb{M}=0, \qquad \mathbb{N}=\frac{-2}{\sqrt{1+4\,u^2+4\,v^2}}. \tag{18}$$

We solve

$$\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}=\Gamma_{11}^1\begin{pmatrix} 1 \\ 0 \\ 2\,u \end{pmatrix}+\Gamma_{11}^2\begin{pmatrix} 0 \\ 1 \\ -2\,v \end{pmatrix}+\frac{2}{1+4\,u^2+4\,v^2}\begin{pmatrix} -2\,u \\ 2\,v \\ 1 \end{pmatrix} \tag{19}$$

to obtain

$$\Gamma_{11}^1=\frac{4\,u}{1+4\,u^2+4\,v^2}, \qquad \Gamma_{11}^2=\frac{-4\,v}{1+4\,u^2+4\,v^2}. \tag{20}$$

Clearly

$$\Gamma_{12}^1=\Gamma_{12}^2=0. \tag{21}$$

We solve

$$\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \Gamma_{22}^1 \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix} + \Gamma_{22}^2 \begin{pmatrix} 0 \\ 1 \\ -2v \end{pmatrix} + \frac{-2}{1+4u^2+4v^2} \begin{pmatrix} -2u \\ 2v \\ 1 \end{pmatrix} \tag{22}$$

to obtain

$$\Gamma_{22}^1 = \frac{-4u}{1+4u^2+4v^2}, \qquad \Gamma_{22}^2 = \frac{4v}{1+4u^2+4v^2}. \tag{23}$$

b) The geodesic equations are

$$\ddot{u} + \frac{4u}{1+4u^2+4v^2} (\dot{u}^2 - \dot{v}^2) = 0, \tag{24}$$

$$\ddot{v} + \frac{4v}{1+4u^2+4v^2} (\dot{v}^2 - \dot{u}^2) = 0. \tag{25}$$

Along a curve $\gamma(t) = \sigma(u(t), 0)$ these become

$$\ddot{u} + \frac{4u}{1+4u^2} \dot{u}^2 = 0, \qquad 0 = 0. \tag{26}$$

We take $\gamma(t)$ to arc length parametrized, that is $(1+4u^2)\,\dot{u}^2 = 1$. Now take $u(t)$ such that $\dot{u} = \frac{1}{\sqrt{1+4u^2}}$. We calculate

$$\begin{aligned} \ddot{u} + \frac{4u}{1+4u^2} \dot{u}^2 &= -\frac{1}{2} \frac{8u}{\sqrt{1+4u^2}^3} \dot{u} + \frac{4u}{1+4u^2} \dot{u}^2 \\ &= \frac{4u\dot{u}}{1+4u^2} \left(-\frac{1}{\sqrt{1+4u^2}} + \dot{u} \right) \\ &= 0. \end{aligned} \tag{27}$$

Thus $v = 0$ can be parametrized into a geodesic.

Remark. In fact, as we have mentioned in class, any two of the three equations (the two geodesic equations, and the equation for constant speed) imply the third. Thus the calculation in (27) is not really necessary.

QUESTION 4. (BONUS 2 PTS) *Let S_1, S_2 be two surfaces that intersect along a curve $\mathcal{C}$. Further assume that S_1 and S_2 are not tangent along $\mathcal{C}$. Prove: If $\mathcal{C}$ can be parametrized into a geodesic of S_1, then it cannot be parametrized into a geodesic of S_2.*

Proof. Let N_1, N_2 be the surface normals of S_1, S_2 along $\mathcal{C}$. Let κ_{1n}, κ_{1g}, κ_{2n}, κ_{2g} be the normal and geodesic curvatures. Let T, N be the curve tangent and normal of $\mathcal{C}$, and let κ be the curve curvature of $\mathcal{C}$. Then

$$\kappa N = \kappa_{1n} N_1 + \kappa_{1g} (N_1 \times T), \tag{28}$$

$$\kappa N = \kappa_{2n} N_2 + \kappa_{2g} (N_2 \times T). \tag{29}$$

As $\mathcal{C}$ can be parametrized into a geodesic on S_1, there must hold $\kappa_{1g} = 0$, that is $N = \pm N_1$. As S_1, S_2 are not tangent, $N = \pm N_2$ does not hold. Consequently $\kappa_{2g} \neq 0$ and the conclusion follows. $\square$