Math 348 Differential Geometry of Curves and Surfaces

Review for Midterm 2

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Please do not hesitate to interrupt me if you have a question.

Concepts

The Fundamental Forms

S: Surface; $p \in S$ a point.

- **The first fundamental form.**
  - A "scale" at p;
  - Includes all information about measuring velocities at p;
  - A bilinear form on T_pS;
  - Three numbers $\mathbb{E}(p), \mathbb{F}(p), \mathbb{G}(p)$:

    $$\langle v, w \rangle_{p,S} = \mathbb{E}(p) v_1 w_1 + \mathbb{F}(p)(v_1 w_2 + v_2 w_1) + \mathbb{G}(p) v_2 w_2.$$
- **The second fundamental form.**
  - Contains all information about how S curves at p;
  - A bilinear form on T_pS;
  - Three numbers $\mathbb{L}(p), \mathbb{M}(p), \mathbb{N}(p)$.

    $$\langle\langle v, w \rangle\rangle_{p,S} = \mathbb{L}(p) v_1 w_1 + \mathbb{M}(p)(v_1 w_2 + v_2 w_1) + \mathbb{N}(p) v_2 w_2.$$

Curvatures

- **Normal curvature** $\kappa_n(p, w)$:
  - How much does S curve at p along the direction w;
  - Every curve in S passing p along the direction w is "forced" to curve this much.
- **Geodesic curvature** κ_g: "Voluntary" curving of a curve in a surface;
- **Principle curvatures** κ_1, κ_2: Maximum and minimum normal curvatures;
- **Principal vectors** t_1, t_2: Directions along which the normal curvature equals one of the principal curvatures; $t_1 \perp t_2$;
- **Mean curvature** H: Average of normal curvatures;
- **Gaussian curvature** K: Limiting ratio of areas of $\mathcal{G}(\Omega) \subset \mathbb{S}^2$ and $\Omega \subset S$.

Parallel Transport

w: Tangent vector field along a curve in a surface S.

- **Covariant derivative**: Horizontal part of the derivative;
- **Parallel along the curve**: Covariant derivative is zero.
- **Christoffel symbols** Γ_{ij}^{k}: Components in the basis σ_u, σ_v of the covariant derivatives of σ_u, σ_v along the coordinate curves $u = const$ and $v = const$.

Geodesics

- Constant speed parametrizations of "straight lines" on a surface.
- Curvature equals normal curvature. $\kappa = |\kappa_{\boldsymbol{n}}|$;
- Vanishing geodesic curvature. $\kappa_{\boldsymbol{g}} = 0$;
- Unit tangent stays parallel. $\nabla_\gamma T = 0$;
- Shortest path connecting two (close enough) points.

Calculation

The Fundamental Forms

S: Surface parametrized by $\sigma(u, v)$

- The first fundamental form.

$$\begin{aligned}\mathbb{E}(u,v) &= \|\sigma_u(u,v)\|^2 = \sigma_u \cdot \sigma_u; \\ \mathbb{F}(u,v) &= \sigma_u(u,v) \cdot \sigma_v(u,v); \\ \mathbb{G}(u,v) &= \|\sigma_v(u,v)\|^2 = \sigma_v \cdot \sigma_v.\end{aligned}$$

- The second fundamental form.

$$\begin{aligned}\mathbb{L}(u,v) &= \sigma_{uu}(u,v) \cdot N(u,v) = -\sigma_u \cdot N_u; \\ \mathbb{M}(u,v) &= \sigma_{uv}(u,v) \cdot N(u,v) = -\sigma_u \cdot N_v = -\sigma_v \cdot N_u; \\ \mathbb{N}(u,v) &= \sigma_{vv}(u,v) \cdot N(u,v) = -\sigma_v \cdot N_v.\end{aligned}$$

Measurements on a Surface

$\langle \cdot, \cdot \rangle_{p,S}$: First fundamental form of the surface. $\mathbb{E}, \mathbb{F}, \mathbb{G}$.

- Arc length:

$$L = \int_a^b \langle \dot{\gamma}, \dot{\gamma} \rangle_{p,S}^{1/2} \mathrm{d}t.$$

- Angle between two vectors:

$$\cos \angle(v, w) = \frac{\langle v, w \rangle_{p,S}}{\langle v, v \rangle_{p,S}^{1/2} \langle w, w \rangle_{p,S}^{1/2}}$$

- Angle between two curves: $p = \gamma_1 \cap \gamma_2, \; v = \dot{\gamma}_1, \; w = \dot{\gamma}_2.$
- Area:

$$A = \int_U \sqrt{\mathbb{E}\mathbb{G} - \mathbb{F}^2} \mathrm{d}u \mathrm{d}v.$$

Make sure you know how to apply these formulas!

Curvatures

- Normal curvature: $\kappa_n(p,w) = \frac{\langle\langle w,w\rangle\rangle_{p,S}}{\langle w,w\rangle_{p,S}}$.
- Geodesic curvature. $\kappa N = \kappa_n N_S + \kappa_g (N_S \times T)$.
- Principal curvatures.

$$\det \begin{pmatrix} \mathbb{L} - \kappa\mathbb{E} & \mathbb{M} - \kappa\mathbb{F} \\ \mathbb{M} - \kappa\mathbb{F} & \mathbb{N} - \kappa\mathbb{G} \end{pmatrix} = 0.$$

- Principal vectors: $t_i = a_i\sigma_u + b_i\sigma_v$ with

$$\left[\begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix} - \kappa_i \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}\right] \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- $H = \frac{\kappa_1+\kappa_2}{2}$, $K = \kappa_1\kappa_2$.
- $A = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}$. $H = \frac{1}{2} TrA$, $K = \det A$.

Christoffel Symbols

N: Surface normal.

$$\begin{aligned} \sigma_{uu} &= \Gamma_{11}^1 \sigma_u + \Gamma_{11}^2 \sigma_v + \mathbb{L} N; \\ \sigma_{uv} &= \Gamma_{12}^1 \sigma_u + \Gamma_{12}^2 \sigma_v + \mathbb{M} N; \\ \sigma_{vv} &= \Gamma_{22}^1 \sigma_u + \Gamma_{22}^2 \sigma_v + \mathbb{N} N. \end{aligned}$$

$uu \leftrightarrow 11$; $uv \leftrightarrow 12$; $vv \leftrightarrow 22$.

Checking Parallel Transport

$w = \alpha\sigma_u + \beta\sigma_v$: Vector field along $\gamma(s) = \sigma(u(s), v(s))$.

$\Gamma_{11}^1, \ldots, \Gamma_{22}^2$: Christoffel symbols.

$$\begin{aligned}
\dot{\alpha} + (\Gamma_{11}^1 \dot{u} + \Gamma_{12}^1 \dot{v})\alpha + (\Gamma_{12}^1 \dot{u} + \Gamma_{22}^1 \dot{v})\beta &= 0; \\
\dot{\beta} + (\Gamma_{11}^2 \dot{u} + \Gamma_{12}^2 \dot{v})\alpha + (\Gamma_{12}^2 \dot{u} + \Gamma_{22}^2 \dot{v})\beta &= 0
\end{aligned}$$

Checking Geodesics

$\mathbb{L}, \mathbb{M}, \mathbb{N}$: 2nd fundamental form; Γ_{ij}^{k}: Christoffel symbols.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{E}\dot{u}+\mathbb{F}\dot{v}) &= \frac{1}{2}(\mathbb{E}_u\dot{u}^2+2\mathbb{F}_u\dot{u}\dot{v}+\mathbb{G}_u\dot{v}^2).\\ \frac{\mathrm{d}}{\mathrm{d}s}(\mathbb{E}\dot{u}+\mathbb{F}\dot{v}) &= \frac{1}{2}(\mathbb{E}_v\dot{u}^2+2\mathbb{F}_v\dot{u}\dot{v}+\mathbb{G}_v\dot{v}^2).\end{aligned}$$

or

$$\begin{aligned}\ddot{u}+\Gamma_{11}^{1}\dot{u}^2+2\Gamma_{12}^{1}\dot{u}\dot{v}+\Gamma_{22}^{1}\dot{v}^2 &= 0;\\ \ddot{v}+\Gamma_{11}^{2}\dot{u}^2+2\Gamma_{12}^{2}\dot{u}\dot{v}+\Gamma_{22}^{2}\dot{v}^2 &= 0.\end{aligned}$$

- Not satisfied if $\gamma(s)=\sigma(u(s),v(s))$ does not have constant speed.
- Constant speed $\Leftrightarrow \mathbb{E}\dot{u}^2+2\mathbb{F}\dot{u}\dot{v}+\mathbb{G}\dot{v}^2$ is constant.

Examples

Measurements

Example

$\sigma(u, v) = (u, v, e^{u+v+v^2})$. Calculate $\cos\theta$ where θ is the angle between $u = v$ and $v = 0$.

1. Parametrize the two curves: $\sigma(t, t), \sigma(t, 0)$.
2. Find the intersection point p: $u = 0, v = 0$;
3. Calculate $\mathbb{E}, \mathbb{F}, \mathbb{G}$ at p:

$$\mathbb{E} = 2, \mathbb{F} = 1, \mathbb{G} = 2.$$

4. Calculate tangent vectors: $\sigma_u + \sigma_v, \sigma_u$, so

$$v_1 = v_2 = 1, \qquad w_1 = 1, w_2 = 0.$$

5. Calculate angle.

Curvatures

Example

$\sigma(u, v) = (u, v, e^{uv})$. Calculate principal, mean, Gaussian curvatures at $p = (0, 0, 1)$.

1. $p = \sigma(0, 0)$;
2. $\mathbb{E} = 1, \mathbb{F} = 0, \mathbb{G} = 1,\ \mathbb{L} = 0, \mathbb{M} = 1, \mathbb{N} = 0$;
3. Solve $\kappa_1 = 1, \kappa_2 = -1$;
4. $H = 0,\ K = -1$.

Christoffel symbols, Parallel transport, Geodesics.

Example

$\sigma(u, v) = (u, v, e^{uv})$. Calculate Γ_{ij}^k along the curve $u = 0$ and check if $u = 0$ can be re-parametrized into a geodesic. .

1. Calculate $\sigma_u, \sigma_v, N, \sigma_{uu}, \sigma_{uv}, \sigma_{vv}$;
2. Solve Γ_{ij}^k along $u = 0$;

$$\Gamma_{11}^1 = \frac{v^3}{1+v^2}, \qquad \Gamma_{12}^1 = \frac{v}{1+v^2}, \qquad \text{all other } \Gamma_{ij}^k = 0.$$

3. Arc length parametrize $u = 0$: (Note $u(t) = 0$)

$$\mathbb{E}\dot{u}^2 + 2\mathbb{F}\dot{u}\dot{v} + \mathbb{G}\dot{v}^2 = 1 \Rightarrow v(t) = t.$$

4. Check geodesic equations.

Looking Back and Forward

Required Textbook Sections

Not everything required is covered in this review,

only the most important topics. You should also:

1. Review the lecture notes;
2. Review the required sections in the textbook:

   6.1; 7.1 – 7.4; 8.1 – 8.2; 9.1–9.4;

3. Review the optional sections in the textbook:

   6.2 – 6.5; 8.3 – 8.6; 9.5;

4. Review Homeworks.

See you Thursday!

Midterm II

- 11:10am – 12:10pm.
- 3 regular problems and 1 bonus problem.
- Total points 15 + 2 or 3 (depending on the difficulty of the bonus problem).
- Geodesic equations will be provided.