Midterm 1

(Oct. 5, 2017, 11:10am–12:10pm. Total 15+2 pts)

NAME: ______________________ ID#: ______________________

- There are three regular problems and one bonus problem (total 15 pts + 2 bonus pts).
- Please write clearly and show enough work.

QUESTION 1. (5 PTS) *Consider the surface S given by $\sigma(u,v)=(u,v,u^2-v^2)$. Let $\mathcal{G}$ be its Gauss Map. Calculate $\mathcal{G}(p)$ at $p=(2,1,3)$.*

Solution. We have $p=\sigma(2,1)$. Furthermore

$$\sigma_u=(1,0,2\,u), \sigma_v=(0,1,-2\,v) \Longrightarrow \sigma_u(2,1)=(1,0,4), \sigma_v(2,1)=(0,1,-2). \tag{1}$$

Thus

$$\mathcal{G}(p)=\frac{(1,0,4)\times(0,1,-2)}{\|(1,0,4)\times(0,1,-2)\|}=\frac{(-4,2,1)}{\|(-4,2,1)\|}=\frac{1}{\sqrt{21}}(-4,2,1). \tag{2}$$

QUESTION 2. (5 PTS) *Consider the surface S given by $\sigma(u, v) = (u, v, u^2 - v^2)$. Let $\mathcal{G}$ be its Gauss Map. Calculate $D_p\mathcal{G}(w)$ where $p = (2, 1, 3)$ and $w = (1, 2, 0)$.*

Solution.

i. $p = (2, 1, 3) = \sigma(2, 1)$;

ii. $\sigma_u = (1, 0, 2\,u), \sigma_v = (0, 1, -2\,v)$. Thus $\sigma_u(2, 1) = (1, 0, 4), \sigma_v(2, 1) = (0, 1, -2)$.

iii. We have $(1, 2, 0) = a\,(1, 0, 4) + b\,(0, 1, -2)$ with $a = 1, b = 2$.

iv. We have

$$N(u, v) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{(-2\,u, 2\,v, 1)}{\sqrt{1 + 4\,u^2 + 4\,v^2}}. \tag{3}$$

v. Therefore

$$N_u = \frac{(-2, 0, 0)}{\sqrt{1 + 4\,u^2 + 4\,v^2}} - \frac{4\,u\,(-2\,u, 2\,v, 1)}{\sqrt{1 + 4\,u^2 + 4\,v^2}^3} \Longrightarrow N_u(2, 1) = \frac{(-10, -16, -8)}{21^{3/2}}, \tag{4}$$

$$N_v = \frac{(0, 2, 0)}{\sqrt{1 + 4\,u^2 + 4\,v^2}} - \frac{4\,v\,(-2\,u, 2\,v, 1)}{\sqrt{1 + 4\,u^2 + 4\,v^2}^3} \Longrightarrow N_v(2, 1) = \frac{(16, 34, -4)}{21^{3/2}}. \tag{5}$$

vi. Finally

$$D_p\mathcal{G}(w) = 21^{-3/2}\,[(-10, -16, -8) + 2\,(16, 34, -4)] = \frac{(22, 52, -16)}{21^{3/2}}. \tag{6}$$

QUESTION 3. (5 PTS) *Find values $a, b \in \mathbb{R}$ such that the curve $\gamma(t) = (a\cos t, a\sin t, b\,t)$ has curvature $\kappa(t) = 1$, and torsion $\tau(t) = -2$. Justify your answer.*

Solution.

- Arc length parametrization.

  We have $\dot{\gamma}(t) = (-a\ \sin\ t,\ a\ \cos\ t,\ b) \Longrightarrow \dot{S}(t) = \|\dot{\gamma}\| = \sqrt{a^2+b^2} \Longrightarrow S(t) = \sqrt{a^2+b^2}\,t \Longrightarrow T(s) = s/\sqrt{a^2+b^2}$. Thus the arc length parametrization is

$$\Gamma(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, b\,\frac{s}{\sqrt{a^2+b^2}}\right). \tag{7}$$

- Calculation. We have

$$\dot{\Gamma}(s) = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\frac{s}{\sqrt{a^2+b^2}}, a\cos\frac{s}{\sqrt{a^2+b^2}}, b\right), \tag{8}$$

$$\ddot{\Gamma}(s) = \frac{1}{a^2+b^2}\left(-a\cos\frac{s}{\sqrt{a^2+b^2}}, -a\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right) \Longrightarrow \kappa(s) = \frac{|a|}{a^2+b^2}. \tag{9}$$

$$\dddot{\Gamma}(s) = \frac{1}{(a^2+b^2)^{3/2}}\left(a\sin\frac{s}{\sqrt{a^2+b^2}}, -a\cos\frac{s}{\sqrt{a^2+b^2}}, 0\right). \tag{10}$$

  We calculate

$$\tau(s) = \frac{\left(\dot{\Gamma}(s)\times\ddot{\Gamma}(s)\right)\cdot\dddot{\Gamma}(s)}{\kappa(s)^2} = \frac{b}{a^2+b^2}. \tag{11}$$

We see that $a = \frac{1}{5}, b = -\frac{2}{5}$ satisfy the requirement.

QUESTION 4. (BONUS. 2 PTS) *Let $\gamma(t)$ be a curve with $\kappa(t) = 2\,\tau(t)$ for all t. Prove that there is a constant vector w_0 such that the angle between w_0 and $B(t)$ is constant.*

Proof. Notice that

$$\frac{\mathrm{d}}{\mathrm{d}s}(T + 2\,B) = \kappa\,N - 2\,\tau\,N = 0. \tag{12}$$

We take $w_0 = T + 2\,B$. Then

$$\cos\angle(B, w_0) = \frac{(T + 2\,B)\cdot B}{\|T + 2\,B\|\,\|B\|} = \frac{2}{\sqrt{5}}, \tag{13}$$

and the angle has to be constant. □