Math 348 Differential Geometry of Curves and Surfaces

Review for Midterm 1

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Table of contents

Required sections: 1.1, 1.2, 2.1–2.3, 4.1–4.5;

Optional sections: 1.3–1.5, 4.5, 5.1–5.6.

Please do not hesitate to interrupt me if you have a question.

Curve Theory

Regular Curve

- **Definition.**
  A map $\gamma : (\alpha, \beta) \mapsto \mathbb{R}^n$, for some $-\infty \leqslant \alpha < \beta \leqslant \infty$, such that $\gamma \in C^\infty$, and $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.
- **Trace.**
  $\mathcal{C} = \gamma((\alpha, \beta))$.
- **Tangent vector.**
  - Tangent vector to the curve $\gamma(t)$ at $t = t_0$: $\dot{\gamma}(t_0)$;
  - Tangent vectors (tangent line) to the trace at $p = \gamma(t_0)$: $\{a\dot{\gamma}(t_0) : a \in \mathbb{R}\}$.
- **Arc length.**

$$L = \int_a^b \|\dot{\gamma}(t)\| \, \mathrm{d}t.$$

Arc Length Parametrization

Given $\gamma(t)$. Find $\Gamma(s)$ with the same trace with $\left\|\dot{\Gamma}(s)\right\| = 1$.

1. Calculate $\|\dot{\gamma}(t)\|$.
2. Solve $\dot{S}(t) = \|\dot{\gamma}(t)\|$.
3. Find the inverse function $T(s)$.
4. $\Gamma(s) = \gamma(T(s))$.

Differential Geometry of Curves with Arc Length Parametrization

- $T(s) = \dot{\gamma}(s)$;
- $N(s) = \frac{\ddot{\gamma}(s)}{\|\ddot{\gamma}(s)\|}$;
- $B(s) = T(s) \times N(s)$;
- $\kappa(s) = \|\ddot{\gamma}(s)\|$;
- $\tau(s) = \frac{(\dot{\gamma}(s) \times \ddot{\gamma}(s)) \cdot \dddot{\gamma}(s)}{\kappa^2(s)}$;
- Frenet-Serret:

$$
\begin{aligned}
\dot{T}(s) &= \kappa(s)N(s), && (1)\\
\dot{N}(s) &= -\kappa(s)T(s) + \tau(s)B(s), && (2)\\
\dot{B}(s) &= -\tau(s)N(s). && (3)
\end{aligned}
$$

Differential Geometry for Curves with General Parametrization

- Required.
  - $T(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$;
  - $B(t) = \frac{\dot{\gamma}(t) \times \ddot{\gamma}(t)}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}$;
  - $N(t) = B(t) \times T(t)$.
- Optional.
  - $\kappa(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3}$;
  - $\tau(t) = \frac{(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \cdot \dddot{\gamma}(t)}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|^2}$.

$\gamma(s) = (x(s), y(s))$: Arc length parametrized plane curve.

- **Signed normal.**
  $N_S(s)$: $T(s)$ rotates $\pi/2$ counter-clockwise;
- **Signed curvature.**
$$\ddot{\gamma}(s) = \kappa_S(s) N_S(s).$$
- **Relation between signed curvature and curvature.**
$$\kappa_S(s) N_S(s) = \kappa(s) N(s).$$

Surface Theory

Surface/Surface Patch

- **Regular surface patch.**
  $\sigma : U \mapsto \mathbb{R}^3$. $U \subseteq \mathbb{R}^2$. σ: bijective. Both σ, σ^{-1} smooth. $\sigma_u \times \sigma_v \neq 0$ anywhere.
- **Tangent plane.**

$$T_{\sigma(u_0,v_0)}S = \{a\sigma_u(u_0, v_0) + b\sigma_v(u_0, v_0) : a, b \in \mathbb{R}\}.$$

- **Unit normal.**

$$N_{\sigma(u_0,v_0)} = N(u_0, v_0) = \frac{\sigma_u(u_0, v_0) \times \sigma_v(u_0, v_0)}{\|\sigma_u(u_0, v_0) \times \sigma_v(u_0, v_0)\|}$$

Make sure you are aware of the sign problem!

- **Surface area.**

$$A = \int_U \|\sigma_u(u, v) \times \sigma_v(u, v)\| \,\mathrm{d}u\mathrm{d}v.$$

Functions Between Surfaces

$$\boxed{S = \sigma(u, v), \tilde{S} = \tilde{\sigma}(\tilde{u}, \tilde{v}), f : S \mapsto \tilde{S}.}$$

- "Helper" function.

$$F = (\tilde{\sigma})^{-1} \circ f \circ \sigma.$$

- **Differential.**
  - Understanding $D_p f(w)$.

    $$D_{\text{point in } S} f(\text{vector in tangent plane of } S) = \text{vector in tangent plane of } \tilde{S}.$$

    $D_p f$ is one function, acting on w!
  - Formula.

    $$D_p f(a\sigma_u + b\sigma_v) = \tilde{a}\tilde{\sigma}_{\tilde{u}} + \tilde{b}\tilde{\sigma}_{\tilde{v}}.$$

    σ_u, σ_v at p, $\tilde{\sigma}_{\tilde{u}}, \tilde{\sigma}_{\tilde{v}}$ at $f(p)$.

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = DF(u, v) \cdot \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \sigma(u, v) = p.$$

Calculating $D_p f(w)$

$$\boxed{\sigma : U \mapsto S,\ \tilde{\sigma} : \tilde{U} \mapsto \tilde{S},\ f : S \mapsto \tilde{S},\ p \in S.}$$

1. Find u_0, v_0 such that $\sigma(u_0, v_0) = p$;
2. Calculate $\sigma_u(u_0, v_0), \sigma_v(u_0, v_0)$;
3. Find a, b such that $w = a\sigma_u(u_0, v_0) + b\sigma_v(u_0, v_0)$;
4. Calculate $DF(u_0, v_0)$ for $F = (\tilde{\sigma})^{-1} \circ f \circ \sigma$;
5. Calculate
$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = DF(u_0, v_0) \cdot \begin{pmatrix} a \\ b; \end{pmatrix}$$
6. Find $\tilde{u}_0, \tilde{v}_0$ such that $\tilde{\sigma}(\tilde{u}_0, \tilde{v}_0) = f(p)$;
7. Calculate $\tilde{\sigma}_u(\tilde{u}_0, \tilde{v}_0), \tilde{\sigma}_v(\tilde{u}_0, \tilde{v}_0)$;
8.
$$D_p f(w) = \tilde{a}\tilde{\sigma}_u(\tilde{u}_0, \tilde{v}_0) + \tilde{b}\tilde{\sigma}_v(\tilde{u}_0, \tilde{v}_0).$$

The Gauss Map

- **Definition.**

$$\mathcal{G} : S \mapsto \mathbb{S}^2, \qquad \mathcal{G}(p) = N(u,v), \qquad p = \sigma(u,v).$$

- **Natural patches.**

$$\tilde{U} = U; \qquad \tilde{\sigma}(u,v) = N(u,v) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}.$$

- **Calculation of $D_p\mathcal{G}(w)$.**
  1. Find $\sigma(u_0, v_0) = p$;
  2. Calculate $\sigma_u(u_0, v_0), \sigma_v(u_0, v_0)$;
  3. Find a, b, $w = a\sigma_u(u_0, v_0) + b\sigma_v(u_0, v_0)$;
  4. Calculate $N(u,v) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$;
  5. Calculate $N_u(u_0, v_0), N_v(u_0, v_0)$;
  6. $$D_p\mathcal{G}(a\sigma_u + b\sigma_v) = aN_u + bN_v.$$

Examples

Curve Problem I: Arc Length. T, N, B, κ, τ

Example

Calculate T, N, B, κ, τ for the curve $\gamma(t) = (\cos t, \sin t, 2t)$.

1. **Arc length parametrization.**

$$\Gamma(s) = (\cos \frac{s}{\sqrt{5}}, \sin \frac{s}{\sqrt{5}}, \frac{2s}{\sqrt{5}}).$$

2. **Apply formulas.**

$$T(s) = \dot{\Gamma}(s) = \frac{1}{\sqrt{5}}(-\sin, \cos, 2); \quad (4)$$

$$N(s) = \frac{\ddot{\Gamma}(s)}{\left\|\ddot{\Gamma}(s)\right\|} = (-\cos, -\sin, 0), \qquad \kappa(s) = \frac{1}{5}; \quad (5)$$

$$B(s) = T(s) \times N(s) = \frac{1}{\sqrt{5}}(2\sin, -2\cos, 1); \quad (6)$$

$$\tau(s) = B(s) \cdot \frac{\dddot{\Gamma}}{\kappa} = \frac{2}{5}. \quad (7)$$

Curve Problpem II: Frenet-Serret

Example

(2016 Mid 1 Q4) Prove

$$[(T \times \dot{T}) \cdot \ddot{T}][(B \times \dot{B}) \cdot \ddot{B}] = \kappa^3 \tau^3.$$

Proof.

- $\dot{T} = \kappa N$; $\ddot{T} = \dot{\kappa} N + \kappa \dot{N} = -\kappa^2 T + \dot{\kappa} N + \kappa \tau B$.

$$(T \times \dot{T}) \cdot \ddot{T} = \kappa^2 \tau.$$

- $\dot{B} = -\tau N$; $\ddot{B} = -\dot{\tau} N - \tau \dot{N} = \tau \kappa T - \dot{\tau} N - \tau^2 B$.

$$(B \times \dot{B}) \cdot \ddot{B} = \kappa \tau^2.$$

- QED.

□

Surface Problem I: Surface Area

Example

Calculate the surface area of the part of S, given by $(u, v, u^2 + v^2)$, between $z = 0$ and $z = 1$.

1. $U = \{(u, v) : u^2 + v^2 < 1\}$;
2. $\sigma_u = (1, 0, 2u), \sigma_v = (0, 1, 2v) \to \sigma_u \times \sigma_v = (-2u, -2v, 1)$;
3. $\|\sigma_u \times \sigma_v\| = \sqrt{1 + 4u^2 + 4v^2}$;
4. $$A = \int_U \sqrt{1 + 4u^2 + 4v^2}\mathrm{d}u\mathrm{d}v = \frac{\pi}{6}(2^{3/2} - 1).$$

Example

(2016 Mid 1 Q3) Let S be given by $(u, v, u^2 + v^2)$. Calculate $D_p\mathcal{G}(w)$ where $p = (1, 1, 2)$, $w = (-1, 1, 0)$.

1. $p = \sigma(1, 1)$;
2. $\sigma_u(1, 1) = (1, 0, 2)$, $\sigma_v(1, 1) = (0, 1, 2)$;
3. $w = -\sigma_u + \sigma_v$, so $D_p\mathcal{G}(w) = -N_u + N_v$;
4. $N(u, v) = \frac{(-2u,-2v,1)}{\sqrt{1+4u^2+4v^2}}$.

$$N_u(1, 1) = \frac{1}{27}(-10, 8, -4), \quad N_v(1, 1) = \frac{1}{27}(8, -10, -4).$$

5. $D_p\mathcal{G}(w) = \frac{1}{3}(2, -2, 0)$.

Without using properties of the Gauss map: See 2016 Midterm 1 Solutions.

Looking Back and Forward

Summary

κ, τ, T, N, B; Frenet-Serret.

Surface area; Functions between surfaces; Gauss Map.

See You Thursday!

1 hour (11:10 - 12:10); 3-4 (+1?) problems.

I am ready. Hope you are too!