Math 348 Differential Geometry of Curves and Surfaces

Lecture 20: The Gauss-Bonnet Theorem II: Gauss-Bonnet on Surfaces

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Please do not hesitate to interrupt me if you have a question.

The Full Gauss-Bonnet Theorem

The Gauss-Bonnet Theorem

Theorem

S: Surface. $\mathcal{C} \subset S$ a simple[1] *closed curve. Ω: The part of S that is enclosed by $\mathcal{C}$. Then*

$$\int_{\Omega} K \mathrm{d}S + \int_{\mathcal{C}} \kappa_g \mathrm{d}s = 2\pi.$$

- Intuition:

$$\text{Total angle of turning} = \text{Forced turn} + \text{Voluntary turn.}$$

[1] Does not intersect itself.

A Paradox

$\boxed{S\text{: unit sphere; } \mathcal{C} \text{ arbitrary circle on } S \text{ centering at the north pole.}}$

- $\mathcal{C}$ divides S into north region Ω_N and south region Ω_S.
- Apply Gauss-Bonnet:

$$\int_{\Omega_N} K + \int_{\mathcal{C}} \kappa_g = \int_{\Omega_S} K + \int_{\mathcal{C}} \kappa_g .$$

- Conclusion

$$\int_{\Omega_N} K = \int_{\Omega_S} K .$$

What did we do wrong?

Proof of Gauss-Bonnet

Setting up

- Parametrize $\mathcal{C}$ by $\gamma(s) = \sigma(u(s), v(s))$;
- $W(s)$: Unit tangent vector of S that is parallel along $\mathcal{C}$;
- $\theta(s)$: Angle between $\dot{\gamma}(s)$ and $W(s)$;
- $N_S(s)$: Surface normal along $\mathcal{C}$.

$$\dot{\gamma}(s) = (\cos\theta(s))W(s) + (\sin\theta(s))W(s) \times N_S(s).$$

Understanding geodesic curvature

$$\begin{aligned}\ddot{\gamma}(s) &= \dot{\theta}(s)[(-\sin\theta(s))W(s) + (\cos\theta(s))W(s)\times N_S(s)]\\ &\quad +(\cos\theta(s))\dot{W}(s) + (\sin\theta(s))\dot{W}(s)\times N_S(s)\\ &\quad +(\sin\theta(s))W(s)\times\dot{N}_S(s).\end{aligned}$$

- Recall

$$\kappa N = \kappa_n N_S + \kappa_g N_S \times T.$$

- Therefore $\kappa_g(s) = \dot{\theta}(s)$.

Consequently

$$\int_{\mathcal{C}} \kappa_g \mathrm{d}s = 2\pi - \Theta.$$

where Θ is the angle from the starting W to the ending W.

$\int K = \Theta$

- σ: geodesic surface patch. $\mathrm{d}u^2 + \mathbb{G}\mathrm{d}v^2$;
- $e_1 = \sigma_u$, $e_2 = \sigma_v/\mathbb{G}^{1/2}$.

$$W(s) = [\cos\theta(s)]e_1 + [\sin\theta(s)]e_2.$$

- $\dot{W} \cdot [(-\sin\theta)e_1 + (\cos\theta)e_2] = 0 \Rightarrow \dot{\theta} = \dot{e}_2 \cdot e_1$;
- $\mathcal{C}' : (u(s), v(s))$, enclosing Ω'. By Green's Theorem,

$$\int_{\mathcal{C}} \dot{e}_2 \cdot e_1 = \int_{\Omega'} [e_{1,u} \cdot e_{2,v} - e_{1,v} \cdot e_{2,u}]\mathrm{d}u\mathrm{d}v.$$

$\int K = \Theta$, cont.

$$\boxed{\Theta = \int \dot{\theta} = \int_{\Omega'} [e_{1,u} \cdot e_{2,v} - e_{1,v} \cdot e_{2,u}] \mathrm{d}u \mathrm{d}v.}$$

- Substitute $e_1 = \sigma_u, e_2 = \sigma_v / \mathbb{G}^{1/2}$, integrand becomes

$$\sigma_{uu} \cdot \frac{\sigma_{vv}}{\mathbb{G}^{1/2}} - \frac{1}{2} \frac{(\sigma_{uu} \cdot \sigma_v)\mathbb{G}_v}{\mathbb{G}^{3/2}} - \frac{\sigma_{uv} \cdot \sigma_{uv}}{\mathbb{G}^{1/2}} + \frac{1}{2} \frac{(\sigma_{uv} \cdot \sigma_v)\mathbb{G}_u}{\mathbb{G}^{3/2}}.$$

- $\mathbb{E} = 1, \mathbb{F} = 0 \Rightarrow$

$$\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{12} = 0, \Gamma^2_{12} = \frac{\mathbb{G}_u}{2\mathbb{G}}, \Gamma^1_{22} = -\frac{\mathbb{G}_u}{2}, \Gamma^2_{22} = \frac{\mathbb{G}_v}{2\mathbb{G}}.$$

- Calculate

$$\int_{\Omega'} [e_{1,u} \cdot e_{2,v} - e_{1,v} \cdot e_{2,u}] \mathrm{d}u \mathrm{d}v = \int_{\Omega'} K \sqrt{\mathbb{E}\mathbb{G} - \mathbb{F}^2} \mathrm{d}u \mathrm{d}v = \int_{\Omega} K.$$

Gauss-Bonnet for Compact Surfaces

The Theorem

$\boxed{S\text{: Compact surface. Then } \int_S K = 2\pi\chi.}$

- $\int_S K$ has to be defined;
- χ: Euler number.

$$\chi = V - E + F.$$

The Euler number

$$\boxed{\chi = V - E + F.}$$

- "Look like a sphere": $\chi = 2$.
    1. Take away one face. "Flatten". F becomes $F - 1$.
    2. Take away or "merge" edges. $V - E + F$ unchanged.
    3. Until we have a polygon.
- "Look like a donut": $\chi = 0$.
- "Look like a 8": $\chi = -2$.
- Do you see the pattern?

Proof

$$\boxed{\int_S K = 2\pi\chi.}$$

1. S becomes a polyhedron through triangularization. With V, E, F.
2. Apply Gauss-Bonnet to each T_i, $i = 1, 2, \ldots, F$.
3. Sum up $\sum_{i=1}^{F}$.

$$\int_{\cup T_i} K + \sum \int_{edges} \kappa_g + \sum \text{exterior angles} = 2F\pi.$$

4. Counting.

$$\sum \text{exterior angles} - 2F\pi = \sum_{i=1}^{V} (E_i - 2)\pi - 2F\pi = -2\pi\chi.$$

Looking Back and Forward

Summary

$\boxed{\text{Required: §13.1; Optional: §13.2–13.8}}$

1. The Gauss-Bonnet Theorem:

Theorem

S: Surface. $\mathcal{C} \subset S$ a simple closed curve. Ω: The part of S that is enclosed by $\mathcal{C}$. Then

$$\int_{\Omega} K \mathrm{d}S + \int_{\mathcal{C}} \kappa_g \mathrm{d}s = 2\pi.$$

2. Gauss-Bonnet for curvilinear polygons.
3. Gauss-Bonnet for compact surfaces.

See you next Tuesday!

$\boxed{\text{Review for final}}$

Please let me know things you want me to cover/solve/discuss.