Lectures 20: The Gauss-Bonnet Theorem II

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we introduce the Gauss-Bonnet theorem. The required section is §13.1. The optional sections are §13.2–§13.8.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. A mechanical point of view

The role of surface curvature can be understood through the following mechanical analogy.

- Curvature = centrifugal force.

  Consider a particle moving along a plane curve $\mathcal{C}$ with unit speed. Then the position of this particle gives the arc length parametrization of $\gamma(s)$. Then the velocity and acceleration are

  $$v(s) = \dot{\gamma}(s), \qquad a(s) = \ddot{\gamma}(s). \tag{1}$$

  If we denote $n_s(s) := [\dot{\gamma}(s)]^{\perp}$, there holds

  $$a(s) = \kappa_s(s)\, n_s(s). \tag{2}$$

  Thus we see that $\kappa_s(s)$ is the "signed" magnitude of force. Consequently

  $$2\,\pi = \int_{\mathcal{C}} \kappa_s(s)\, \mathrm{d}s = \text{"signed total" of work done.} \tag{3}$$

- Surface curvature = "Gravity" = "extra" centrifugal force.

  Now consider a particle moving along a surface curve. Then part of the the centrifugal force is provided by "gravity"–the force that keeps the particle on the surface. Thus we conjecture that

  $$2\,\pi = \text{"signed total" of work by gravity} + \text{"signed total" of work by other forces.} \tag{4}$$

  Recall that on a surface, the trajectory of a particle moving under gravity only satisfies $\kappa_g = 0$ where κ_g is the geodesic curvature. On the other hand, the total work done by gravity should be related to the "total mass" enclosed by the curve. Thus we reach

  $$2\,\pi = \int_{\Omega} \text{curvature}\, \mathrm{d}S + \int_{\mathcal{C}} \kappa_g(s)\, \mathrm{d}s \tag{5}$$

  where Ω is the part of the surface enclosed by $\mathcal{C}$.

- Geodesic curvature = other forces.

  Exercise 1. Let S be a developable surface. Let $\mathcal{C}$ be a curve on S. Let $\tilde{\mathcal{C}}$ be the curve on the plane that is the "flattened" S. Prove that for any $p \in S$ with $\tilde{p}$ the corresponding point on the plane, there holds $\kappa_g(p) = \kappa_s(\tilde{p})$.

Remark 1. There are other physical explanations for Gauss-Bonnet, for example see here. A more detailed version can be found in A "bicycle wheel" proof of the Gauss-Bonnet theorem, Mark Levi, Expo. Math. 12 (1994), 145–164.

1.1. Gauss-Bonnet on surfaces

THEOREM 2. *Let S be a surface and $\mathcal{C} \subset S$ be a simple closed curve. Let Ω be the part of S that is enclosed by $\mathcal{C}$. There holds*

$$\int_{\Omega} K\, \mathrm{d}S + \int_{\mathcal{C}} \kappa_g\, \mathrm{d}s = 2\,\pi. \tag{6}$$

Exercise 2. Let S be the unit sphere. Let $\mathcal{C} \subset S$ be an arbitrary simple closed curve. Then $\mathcal{C}$ divides S into two regions Ω_N, Ω_S. By Theorem 2 we have

$$\int_{\Omega_N} K \,\mathrm{d}S + \int_{\mathcal{C}} \kappa_g \,\mathrm{d}s = 2\pi = \int_{\Omega_S} K \,\mathrm{d}S + \int_{\mathcal{C}} \kappa_g \,\mathrm{d}s \Longrightarrow \int_{\Omega_N} K \,\mathrm{d}S = \int_{\Omega_S} K \,\mathrm{d}S \tag{7}$$

which means $\operatorname{area}(\Omega_N) = \operatorname{area}(\Omega_S)$. This is absurd. Did we make a mistake?

Proof. We divide the proof of Theorem 2 into several steps.

i. Set-up. We parametrize $\mathcal{C}$ as $\gamma(s) = \sigma(u(s), v(s))$ where s is the arc length parameter. Let the range of s be from 0 to L. Denote by $W(s)$ a parallel tangent unit vector field along $\mathcal{C}$. Let $\theta(s)$ be the angle between $\dot{\gamma}(s)$ and $W(s)$.

Let $N_S(s)$ be the unit normal of S. Then we see that $W(s), N_S(s), W(s) \times N_S(s)$ form a right-handed orthonormal frame, and consequently

$$\dot{\gamma}(s) = (\cos\theta(s))\, W(s) + (\sin\theta(s))\, W(s) \times N_S(s). \tag{8}$$

ii. The role of κ_g. Taking derivative of (8) we have

$$\begin{aligned}\ddot{\gamma}(s) &= \dot{\theta}(s)\, [(-\sin\theta(s))\, W(s) + (\cos\theta(s)\,) W(s) \times N_S(s)] \\ &\quad + (\cos\theta(s))\, \dot{W}(s) + (\sin\theta(s))\, \dot{W}(s) \times N_S(s) \\ &\quad + (\sin\theta(s))\, W(s) \times \dot{N}_S(s).\end{aligned} \tag{9}$$

As $W(s)$ is parallel along $\mathcal{C}$, we see that the black terms are tangent to $T_p S$, the grey term is zero, and the green terms are parallel to $N_S(s)$. Recalling the definition of the normal and geodesic curvatures, we see that

$$\kappa_g(s) = \dot{\theta}(s). \tag{10}$$

Consequently, we have

$$\int_{\mathcal{C}} \kappa_g \,\mathrm{d}s = 2\pi - \Theta \tag{11}$$

where Θ is the angle between $W(0)$ and $W(L)$.

iii. The role of K. Due to the presence of the surface curvature, we do not always have $W(0) = W(L)$, that is $\Theta = 0$, in (11).

We take $\sigma(u,\ v)$ to be a geodesic surface patch, with first fundamental form $\mathrm{d}u^2 + \mathbb{G}(u,v)\,\mathrm{d}v^2$. Let $e_1 := \sigma_u, e_2 := \frac{\sigma_v}{\mathbb{G}^{1/2}}$. Then we have

$$W(s) = [\cos\theta(s)]\, e_1 + [\sin\theta(s)]\, e_2. \tag{12}$$

This leads to

$$\dot{W}(s) = \dot{\theta}(s)\, [(-\sin\theta)\, e_1 + (\cos\theta)\, e_2] + (\cos\theta)\, \dot{e}_1 + (\sin\theta)\, \dot{e}_2. \tag{13}$$

As W is parallel along γ and is of unit length, we have $\dot{W} \perp [(-\sin\theta)\, e_1 + (\cos\theta)\, e_2]$. Thus

$$\begin{aligned} 0 &= \dot{W}(s) \cdot [(-\sin\theta)\, e_1 + (\cos\theta)\, e_2] \\ &= \dot{\theta}(s) + [(\cos\theta)^2\, \dot{e}_1 \cdot e_2 - (\sin\theta)^2\, \dot{e}_2 \cdot e_1] \\ &= \dot{\theta}(s) - \dot{e}_2 \cdot e_1. \end{aligned} \tag{14}$$

Note that we have used $\dot{e}_1 \cdot e_2 = -\dot{e}_2 \cdot e_1$.

Now we have, setting $\mathcal{C}'$ to be the closed plane curve $(u(s), v(s))$ and Ω' the region enclosed by $\mathcal{C}'$, by Green's Theorem,

$$\begin{aligned} \int_0^L e_1 \cdot \dot{e}_2 &= \int_0^L (e_1 \cdot e_{2,u})\, \dot{u} + (e_1 \cdot e_{2,v})\, \dot{v} \\ &= \int_{\mathcal{C}'} (e_1 \cdot e_{2,u})\, \mathrm{d}u + (e_1 \cdot e_{2,v})\, \mathrm{d}v \\ &= \int_{\Omega'} [e_{1,u} \cdot e_{2,v} - e_{1,v} \cdot e_{2,u}]\, \mathrm{d}u\, \mathrm{d}v. \end{aligned} \tag{15}$$

Substituting $e_1 := \sigma_u, e_2 := \frac{\sigma_v}{\mathbb{G}^{1/2}}$ into the above, we have the integrand to be

$$\sigma_{uu} \cdot \frac{\sigma_{vv}}{\mathbb{G}^{1/2}} - \frac{1}{2} \frac{(\sigma_{uu} \cdot \sigma_v)\, \mathbb{G}_v}{\mathbb{G}^{3/2}} - \frac{\sigma_{uv} \cdot \sigma_{uv}}{\mathbb{G}^{1/2}} + \frac{1}{2} \frac{(\sigma_{uv} \cdot \sigma_v)\, \mathbb{G}_u}{\mathbb{G}^{3/2}}. \tag{16}$$

As $\mathbb{E} = 1, \mathbb{F} = 0,$ we have

$$\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = 0, \Gamma_{12}^2 = \frac{\mathbb{G}_u}{2\,\mathbb{G}}, \Gamma_{22}^1 = -\frac{\mathbb{G}_u}{2}, \Gamma_{22}^2 = \frac{\mathbb{G}_v}{2\,\mathbb{G}}. \tag{17}$$

Consequently

$$\sigma_{uu} \cdot \sigma_{vv} = \mathbb{L}\, \mathbb{N}, \qquad \sigma_{uu} \cdot \sigma_v = 0, \tag{18}$$

$$\sigma_{uv} \cdot \sigma_{uv} = \frac{1}{4} \frac{\mathbb{G}_u^2}{\mathbb{G}^2} + \mathbb{M}^2, \qquad \sigma_{uv} \cdot \sigma_v = \frac{\mathbb{G}_u}{2\,\mathbb{G}}. \tag{19}$$

We see that

$$e_{1,u} \cdot e_{2,v} - e_{1,v} \cdot e_{2,u} = \frac{\mathbb{L}\, \mathbb{N} - \mathbb{M}^2}{\mathbb{G}^{1/2}} = K \sqrt{\mathbb{E}\, \mathbb{G} - \mathbb{F}^2}. \tag{20}$$

Therefore

$$\int_{\Omega'} [e_{1,u} \cdot e_{2,v} - e_{1,v} \cdot e_{2,u}]\, \mathrm{d}u\, \mathrm{d}v = \int_{\Omega} K\, \mathrm{d}S \tag{21}$$

and the proof ends. □

Remark 3. The proof of Theorem 2 here is not fully rigorous (can you spot the gaps?). Yet it is intuitive and consistent with our proof in the plane case.

Exercise 3. Read through the proof in §13.1 of the textbook and understand every detail.

Remark 4. By (6) it is easy to see that if $\mathcal{C}$ is a closed geodesic, then necessarily $\int_\Omega K\, \mathrm{d}S = 2\,\pi$. Consequently there is no closed geodesic on a surface with $K \leqslant 0$ everywhere.

Exercise 4. Let S be a cylinder. Then clearly there are closed geodesics. Can you explain this?

THEOREM 5. (CURVILINEAR POLYGONS ON A SURFACE) *For a curvilinear polygon on a surface S, we have*

$$2\,\pi = \int_{\mathcal{C}} \kappa_g\, \mathrm{d}s + \sum \alpha_i + \int_\Omega K\, \mathrm{d}S \tag{22}$$

where α_i are the exterior angles at the vertices.[1]

2. Gauss-Bonnet on compact surfaces

2.1. Integration on compact surfaces

- Recall that we can integrate on a surface patch through

$$\int_S f\,\mathrm{d}S = \int_U f(\sigma(u,v))\,\|\sigma_u \times \sigma_v\|\,\mathrm{d}u\,\mathrm{d}v. \tag{23}$$

What if the surface cannot be covered by one single surface patch? In particular, how do we integrate on a compact surface S?

Exercise 5. Show that a compact surface cannot be covered by one single surface patch.

- The idea is "partition of unity". Assume that S is covered by N surface patches $\sigma_1, \ldots, \sigma_N$, where $\sigma_i: U_i \mapsto S$ with $\Omega_i = \sigma(U_i)$. Note that each Ω_i is open and $\cup_{i=1}^N \Omega_i = S$.

For every Ω_i, let $\tilde{\Omega}_i := \Omega_i - \cup_{j\neq i}\Omega_j$. Then $\tilde{\Omega}_i$ is closed. Let $\sigma(\tilde{U}_i) = \tilde{\Omega}_i$. We see that $\varepsilon_i := \mathrm{dist}(\tilde{U}_i, \partial U_i)/3 > 0$.[2] We define $W_i := \{x \in U_i | \mathrm{dist}(x, \tilde{U}_i) \leqslant \varepsilon_i\}$ and $\tilde{W}_i := \{x \in U_i | \mathrm{dist}(x, \tilde{U}_i) \leqslant 2\,\varepsilon_i\}$.

Next take a smooth even function $\rho \geqslant 0$ such that

$$2\pi \int_0^\infty \rho(t)\,t\,\mathrm{d}t = 1, \qquad \rho(t) = \begin{cases} 1 & |t| < 1/4 \\ 0 & |t| > 3/4 \end{cases}. \tag{24}$$

We see that the function $\phi(u,v) := \rho\left(\sqrt{u^2+v^2}\right)$ satisfies

$$\int_{\mathbb{R}^2} \phi(u,v)\,\mathrm{d}u\,\mathrm{d}v = 2\pi \int_0^\infty \rho(r)\,r\,\mathrm{d}r = 1 \tag{25}$$

and $\phi(u,v) = \begin{cases} 1 & \sqrt{u^2+v^2} < 1/4 \\ 0 & \sqrt{u^2+v^2} > 3/4 \end{cases}$. Now define

$$\phi_i(u,v) := \frac{1}{\varepsilon_i^2}\,\phi\left(\frac{u}{\varepsilon_i}, \frac{v}{\varepsilon_i}\right). \tag{26}$$

Let the function $\chi_i(u,v) := \begin{cases} 1 & (u,v) \in W_i \\ 0 & (u,v) \notin W_i \end{cases}$. Define

$$\Phi_i(u,v) := \int_{\mathbb{R}^2} \phi_i(u-u', v-v')\,\chi_i(u',v')\,\mathrm{d}u'\,\mathrm{d}v'. \tag{27}$$

Then $\Phi_i(u,v)$ is smooth and satisfy

$$\Phi_i(u,v) = \begin{cases} 1 & (u,v) \in \tilde{U}_i \\ >0 & (u,v) \in \tilde{W}_i \\ =0 & (u,v) \text{ outside } \tilde{W}_i \end{cases}. \tag{28}$$

1. Note that our α_i here are different from those in §13.2 of the textbook.
2. If $U_i = \mathbb{R}^2$ just set $\varepsilon_i = 1$.

Finally define $\Psi_i = \frac{\Phi_i \circ \sigma_i^{-1}}{\sum_{j=1}^N \Phi_j \circ \sigma_j^{-1}}$. We see that

$$\sum_{i=1}^N \Psi_i = 1 \text{ all over } S. \tag{29}$$

Such $\{\Psi_i\}$ is called a "partition of unity" of S.

- With such "partition of unity" available, we can simply define

$$\int_S f \, \mathrm{d}S := \sum_{i=1}^N \int_{U_i} F_i(\sigma_i(u,v)) \, \|\sigma_{i,u} \times \sigma_{i,v}\| \, \mathrm{d}u \, \mathrm{d}v \tag{30}$$

where

$$F_i = f \, \Psi_i. \tag{31}$$

2.2. Euler number

DEFINITION 6. *Let P be a polyhedron. Define the* Euler number *χ as*

$$\chi = V - E + F \tag{32}$$

where V is the number of vertices, E the number of edges, and F the number of faces.

Remark 7. It turns out that χ is a "topological invariant". It is easy to convince ourselves that deforming a polyhedron would not change χ. Thus χ depends only on the "shape" of the polygon. A few examples.

- If we can "blow up" the polygon into a sphere, then $\chi = 2$. In other words, $\chi(\text{sphere}) = 2$.

  To see this, we do the following operations.

  i. Take away one face and "flatten" the the "polytope with a hole". Thus $F \mapsto F - 1$ and E, V remain the same.

  ii. Let e be any edge that is not on the boundary. There are two situations.

     a) Both ends of e have more than two edges connected to the vertices. In this case we take e away. then the two adjacent faces are merged together. Thus after this operation we have $E \mapsto E - 1$ and $F \mapsto F - 1$, while V remains the same.

     b) One or both ends of e is connected to one other edge. Then we merge this edge with e. When there is only one such end, the result is $E \mapsto E - 1, V \mapsto V - 1$, when there are two such ends, we have $E \mapsto E - 2$, $V \mapsto V - 2$.

     Note that in either case, $V - E + F$ stays unchanged.

  iii. Keep doing step ii until there is no interior edge anymore. Then we would have a polygon, for which $V = E, F = 1$. Thus the original χ should be $1 + 1 = 2$.

- $\chi(\text{torus}) = 0$.

  The key difference here is that we still cannot flatten the "polytope with a hole" after "taking one face away". Intuitively, if we "cut" the torus and "straighten" it into a cylinder, then $V - E$ stays the same while $F \mapsto F + 2$. But a cylinder (of finite height) is topologically equivalent to the sphere so

$$V + F + 2 - E = 2 \Longrightarrow \chi = V - E + F = 0 \tag{33}$$

  for the torus.

- $\chi(\text{two torus connected together}) = -2$.

2.3. Gauss-Bonnet on compact surfaces

Theorem 8. [3] *Let S be a compact surface. Then*

$$\int_S K \, \mathrm{d}S = 2\,\pi\,\chi. \tag{34}$$

Remark 9. If S is an apple, then the total Gaussian curvature is $2\,\pi$. Now take a pen to poke it. During the process the total Gaussian curvature stays $2\,\pi$. But the moment you poke it through, it becomes 0.

Exercise 6. Explain how does all the Gaussian curvature "disappear" at the moment we "poke through" the apple.

Proof. We sketch the idea. Intuitively, we can divide S into finitely many triangles $T_1, \ldots, T_F$ and thus S becomes a "curvilinear polyhedron" with F faces. We note that as each face has three edges and each edge is shared by two faces, there holds $E = \frac{3\,F}{2}$. On each triangle we apply Theorem 5:

$$\int_{T_i} K \, \mathrm{d}S + \int_{e_{i1} \cup e_{i2} \cup e_{i3}} \kappa_g \, \mathrm{d}s + \alpha_{i1} + \alpha_{i2} + \alpha_{i3} = 2\,\pi \tag{35}$$

where e_{i1}, e_{i2}, e_{i3} are the three edges and α_{i1}, α_{i2}, α_{i3} are the three exterior angles. Now summing over $i = 1, 2, \ldots, F$ we see that[4]

$$\sum_i \int_{T_i} K \, \mathrm{d}S = \int_S K \, \mathrm{d}S, \qquad \sum_i \int_{e_{i1} \cup e_{i2} \cup e_{i3}} \kappa_g \, \mathrm{d}s = 0. \tag{36}$$

Now we sum up the exterior angles through a different way of counting. Let $A_1, \ldots, A_V$ be the vertices. Denote by E_i the number of edges connected to each A_i. Then we see that,

$$\text{sum of exterior angles at } A_i = E_i\,\pi - \sum \text{interior angles at } A_i = (E_i - 2)\,\pi. \tag{37}$$

3. Theorem 13.4.5 of the textbook.

4. Intuitively, we can simply take the edges to be geodesics, then $\kappa_g = 0$ and the edge terms vanish.

Therefore (note that each edge connects two vertices)

$$\begin{aligned}\sum \text{all exterior angles} - 2\,F\pi &= \sum_{i=1}^{V} (E_i - 2)\,\pi - 2\,F\pi \\ &= \left(\sum_{i=1}^{V} E_i\right)\pi - 2\,V\pi - 2\,F\pi \\ &= 2\,E\pi - 2\,V\pi - 2\,F\pi \\ &= 2\,\pi\,(E - V - F) = -2\,\pi\,\chi. \end{aligned} \tag{38}$$

The conclusion then follows. □

Remark 10. We see that the Gaussian curvature is invariant under local isometries, but the integral of the Gaussian curvature over a compact surface is even more invariant–it only depends on the "shape" of the surface.