Math 348 Differential Geometry of Curves and Surfaces

Lecture 19: The Gauss-Bonnet Theorem I: Gauss-Bonnet in the Plane

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Please do not hesitate to interrupt me if you have a question.

The Gauss-Bonnet Theorem

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Theorem

S: Surface. $\mathcal{C} \subset S$ a simple[1] closed curve. Ω: The part of S that is enclosed by $\mathcal{C}$. Then

$$\int_{\Omega} K \mathrm{d}S + \int_{\mathcal{C}} \kappa_g \mathrm{d}s = 2\pi.$$

- Intuition:

$$\text{Total angle of turning} = \text{Forced turn} + \text{Voluntary turn}.$$

- Generalizing "sum of interior angles of a triangle is π".

[1] Does not intersect itself.

Proof of Gauss-Bonnet in the Plane

Gauss-Bonnet for Convex Polygons

$\boxed{\text{Sum of exterior angles of a convex polygon is } 2\pi.}$

- Definition of "exterior angles";
- **Proof.**
    - "Move" all angles to the center; Or,
    - Calculate interior angles.

Gauss-Bonnet for General Polygons

$\boxed{\text{Sum of signed exterior angles of a general polygon is } 2\pi.}$

- Definition of "signed" exterior angle.
- **Proof.** By induction.
    1. Base ($n = 2, 3$) is trivial;
    2. Induction step.

Gauss-Bonnet for Smooth Curves

$$\boxed{\mathcal{C}\text{: Simple closed curve. } \int_{\mathcal{C}} \kappa_s(s)\mathrm{d}s = 2\pi.}$$

- What if not simple?
- Recall "signed curvature".
- **Proof.**
  1. $\kappa_s(s) = \dot{\theta}(s)$;
  2. $\int_{\mathcal{C}} \kappa_s(s)\mathrm{d}s = 2k\pi$ for some integer k.
  3. Approximate by polygon to prove $k = 1$.

Gauss-Bonnet for Curvilinear Polygons

$$\boxed{\int_{\mathcal{C}} \kappa_s \mathrm{d}s + \sum \alpha_i = 2\pi.}$$

- α_i: signed exterior angle;
- Relation between curvature and angle;
- **Proof.** Adapt proof for smooth curves.

Looking Back and Forward

Summary

Required: §13.1; Optional: §13.2–13.8

1. The Gauss-Bonnet Theorem:

Theorem

S: Surface. $\mathcal{C} \subset S$ a simple closed curve. Ω: The part of S that is enclosed by $\mathcal{C}$. Then

$$\int_{\Omega} K \mathrm{d}S + \int_{\mathcal{C}} \kappa_g \mathrm{d}s = 2\pi.$$

See you Thursday!

$\boxed{\text{The Gauss-Bonnet Theorem}}$

- Proof of the Gauss-Bonnet Theorem;
- Relation to the shape of closed surfaces.