Lectures 19: The Gauss-Bonnet Theorem I

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

In this lecture we introduce the Gauss-Bonnet theorem. The required section is §13.1. The optional sections are §13.2–§13.8.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. Gauss-Bonnet for Plane Polygons

Theorem 1. (Gauss-Bonnet for plane triangles) *Let* ABC *be a triangle in the flat plane. Then* $\angle A + \angle B + \angle C = \pi$.

Theorem 2. (Gauss-Bonnet for plane convex polygons) *Let* $A_1 A_2 \ldots A_k$ *be a* k*-polygon in a plane. Further assume that it is convex. Then the sum of its exterior angles is* 2π.

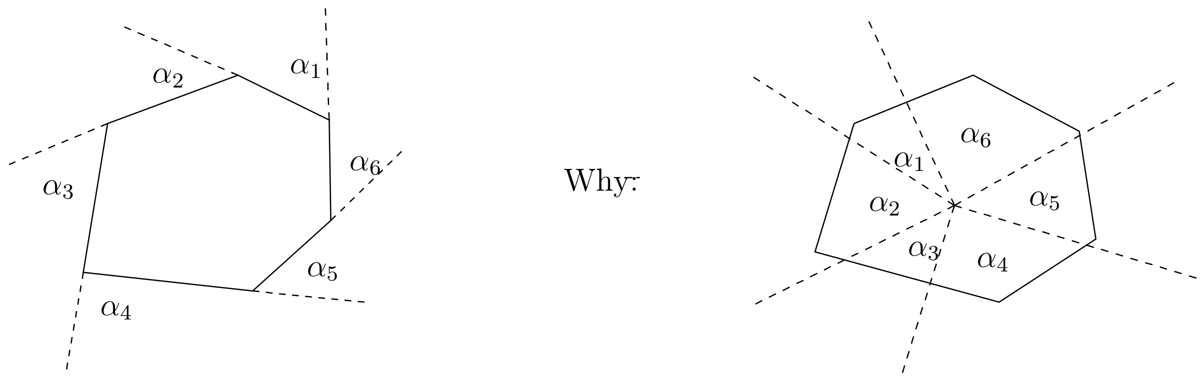


Figure 1. Sum of exterior angles of a convex polygon: $\alpha_1 + \cdots + \alpha_6 = 2\pi$

Remark 3. (Signed exterior angles) The exterior angles of a plane polygon can be signed. Assume that we are traveling along the boundary counterclockwise. Then if the tangent vector is also turning counterclockwise, we say the exterior angle is positive, otherwise it's negative.

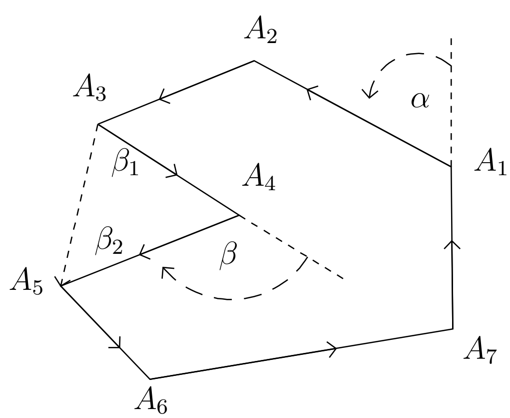


Figure 2. Positive and negative exterior angles: $\alpha > 0$, $\beta < 0$.

Theorem 4. (Gauss-Bonnet for general plane polygons) *The sum exterior angles of a plane polygon, convex or not, is* 2π.

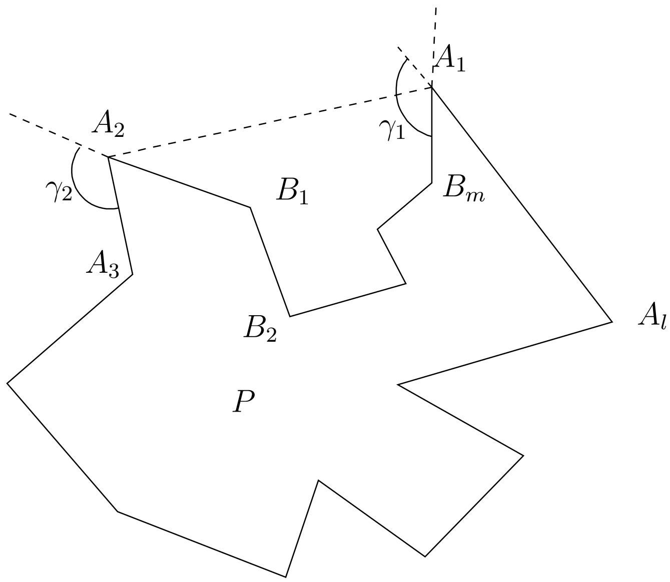
Remark 5. Such a general polygon is a convex polygon (its convex hull) with one or more convex polygon "taken away".

Take the polygon in Figure 2 as an example. Applying Gauss-Bonnet to the small triangle $A_3A_4A_5$ we see that $\beta_1+\beta_2=\beta$ and therefore the sum of exterior angles of the non-convex polygon $A_1\ldots A_7$ is the same as that of the convex polygon $A_1A_2A_3A_5A_6A_7$.

Proof. We prove through induction on the number of vertices n.

- Base. When $n=2,3$ the result is trivial.
- Induction. Assume that the result holds for all $n\leqslant k$. Let P be a $(k+1)$-polygon that is not convex. Then there is a pair of vertices A_1, A_2 of P such that the line segment A_1A_2 is not an edge of P, and the whole P lies to one side of A_1A_2. Like in the figure below.

  Now it is clear that $P=P_A-P_B$ where P_A is the polygon $A_1A_2A_3\cdots A_l$, and P_B is the polygon $A_1A_2B_1\cdots B_m$. It is easy to see that both P_A and P_B have $\leqslant k$ vertices.

  Now we denote the signed exterior angles at $B_1,\ldots,B_m$ by $\beta_1,\ldots,\beta_m$ and the signed exterior angles at $A_1,A_2,A_3,\ldots,A_l$ by $\alpha_1,\ldots,\alpha_l$. By induction hypothesis we have

  $$\alpha_1+\cdots+\alpha_l=\delta_1+\delta_2+\beta_1+\cdots+\beta_m=2\pi. \tag{1}$$

  Here δ_1,δ_2 are the two exterior angles at A_1,A_2, but for the polygon P_B. Next notice that the exterior angles of P are $\alpha_3,\ldots,\alpha_l,-\beta_1,\ldots,-\beta_m$ and two other angles γ_1,γ_2. We thus have

  $$\begin{aligned}\sum \text{exterior angles of } P &= \gamma_1+\gamma_2-\beta_1-\cdots-\beta_m+\alpha_3+\cdots+\alpha_l\\ &= (\gamma_1-\alpha_1+\delta_1)+(\gamma_2-\alpha_2+\delta_2)\\ &= \pi+\pi=2\pi. \end{aligned}\tag{2}$$

  □

2. Gauss-Bonnet for plane curvilinear polygons

- Angle = "concentrated curvature".

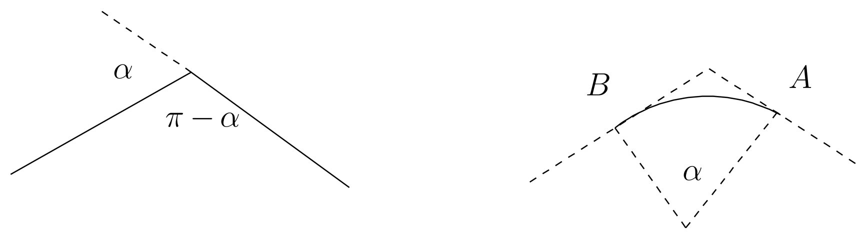


Figure 3. Angle as "concentrated curvature"

Exercise 1. We "smooth" the angle through an arc (part of a circle) that is tangent to the two "arms" at A, B respectively. Prove that

$$\int_A^B \kappa\, \mathrm{d}s = \alpha \tag{3}$$

where the integral is along the arc.

Exercise 2. What if we "smooth" the angle through a different family of arcs, say parabolas?

- Signed curvature for plane curves.

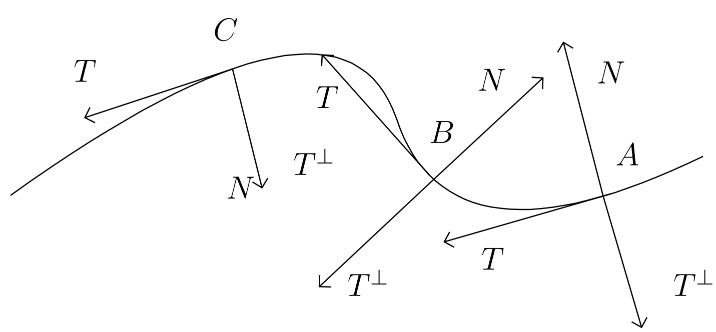


Figure 4. Signed curvature κ_s.

For the plane vector $T = (T_x, T_y)$, we can define $T^{\perp} := (-T_y, T_x)$ which is the counterclockwise rotation of T by $\pi/2$. Clearly $T^{\perp} \perp T$. On the other hand, we have $N = \kappa^{-1} \frac{\mathrm{d}T}{\mathrm{d}s} \perp T$. Thus either $N = T^{\perp}$ or $N = -T^{\perp}$. In the former case we define the signed curvature $\kappa_s = \kappa$ while in the latter case we define $\kappa_s = -\kappa$. For example in Figure 4 $\kappa_s = \kappa$ at C but $= -\kappa$ at A, B.

Exercise 3. Prove that $\kappa = |\kappa_s|$.

Theorem 6. (Gauss-Bonnet for simple closed plane curves) *A smooth, closed, non-self-intersecting planar curve $\mathcal{C}$ satisfies*

$$\int_{\mathcal{C}} \kappa_s(s)\, \mathrm{d}s = 2\,\pi. \tag{4}$$

Proof.

i. We first prove the following. Let the parametrization of $\mathcal{C}$ by arc length be $\gamma(s)$: $[a, b] \mapsto \mathbb{R}^2$. Then

$$\int_a^b \kappa_s(s)\,\mathrm{d}s = \theta_0 + 2\,k\,\pi \tag{5}$$

for some $k \in \mathbb{Z}$.

To see this, notice that we can denote

$$T(s) = \dot{\gamma}(s) = (\cos\theta(s), \sin\theta(s)) \tag{6}$$

where $\theta(s)$ is the angle between $\dot{\gamma}(s)$ and $\dot{\gamma}(a)$.[1] Taking derivative we have

$$\ddot{\gamma}(s) = \dot{\theta}(s)\,(-\sin\theta(s), \cos\theta(s)) = \dot{\theta}(s)\,T^{\perp}(s). \tag{7}$$

Therefore $\kappa_s(s) = \dot{\theta}(s)$. Consequently

$$\int_a^b \kappa_s(s)\,\mathrm{d}s = \theta(b). \tag{8}$$

Since $T(b) = (\cos\theta(b), \sin\theta(b)) = (\cos\theta_0, \sin\theta_0)$, (5) follows.

ii. Thus we see that for a closed simple curve there holds

$$\int_{\mathcal{C}} \kappa_s\,\mathrm{d}s = 2\,k\,\pi \tag{9}$$

for some $k \in \mathbb{Z}$. Now we prove that $k = 1$.

Wlog the parametrization is counterclociwise, that is the point $\gamma(s)$ moves counterclockwise with respect to the interior of $\mathcal{C}$ as s increases. Pick $s_0 = a < s_1 < \cdots < s_k < b = s_{k+1}$ such that

$$\int_{s_{i-1}}^{s_{i+1}} \kappa(s)\,\mathrm{d}s < \pi, \tag{10}$$

for all i. Then for any $s', s'' \in (s_{i-1}, s_{i+1})$, we have

$$\left| \int_{s'}^{s''} \kappa_s(s)\,\mathrm{d}s \right| \leqslant \int_{s'}^{s''} \kappa(s)\,\mathrm{d}s < \pi. \tag{11}$$

On the other hand, we have

$$\int_{s'}^{s''} \kappa_s(s)\,\mathrm{d}s = \delta\theta' + 2\,k\,\pi \tag{12}$$

where $\delta\theta' \in (-\pi, \pi]$ is the angle between $T(s'')$ and $T(s')$.

1. Rigorously speaking, it is the angle between $x'(a)$ and the vector that is the parallel transported $x'(s)$.

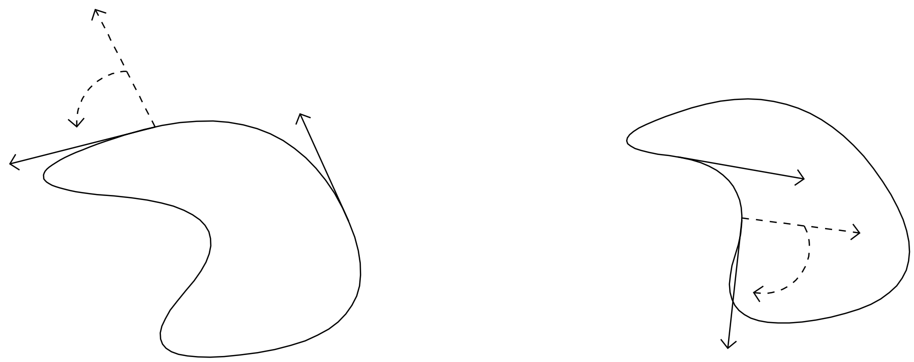


Figure 5. Left: $\delta\theta' > 0$; Right: $\delta\theta' < 0$.

Noticing that $|\delta\theta' + 2k\pi| > \pi$ for all $k \neq 0$, we conclude that

$$\int_{s'}^{s''} \kappa_s(s)\,\mathrm{d}s = \delta\theta', \tag{13}$$

when s', s'' are close enough.

Now let P be the polygon with vertices $\gamma(s_0), \gamma(s_1), \ldots, \gamma(s_k), \gamma(s_{k+1}) = \gamma(s_0)$. Note that P can be made not self-intersecting (see remark below).

By the mean value theorem[2] there are $s_0' \in [s_0, s_1], s_1' \in [s_1, s_2], \ldots, s_k' \in [s_k, s_{k+1}]$ such that $\dot{\gamma}(s_i') \parallel \overrightarrow{\gamma(s_i)\gamma(s_{i+1})}$. Thanks to the arguments (10)–(13) we see that

$$\int_{s_i'}^{s_{i+1}'} \kappa_s(s)\,\mathrm{d}s = \text{the exterior angle at } \gamma(s_i). \tag{14}$$

Now the conclusion (4) follows from Theorem 4, the the polygon Gauss-Bonnet theorem $\square$

Remark 7. In the above proof, besides (10), we also require the partition $s_0, \ldots, s_k$ be such that the polygon P is simple, that is does not intersect itself.

QUESTION 8. *Can this always be done through making $s_{i+1} - s_i$ small enough? If not, can we prove Gauss-Bonnet for polygons that intersect itself?*

THEOREM 9. (GAUSS-BONNET FOR CURVILINEAR POLYGONS) *Let $\alpha_1, \ldots, \alpha_k$ be the exterior angles. Then*

$$\int_{\mathcal{C}} \kappa_s\,\mathrm{d}s + \sum_{i=1}^{k} \alpha_i = 2\pi. \tag{15}$$

Proof. Left as exercise. $\square$

2. Keep in mind that the mean value theorem does not hold for space curves.