Math 348 Differential Geometry of Curves and Surfaces

Lecture 17: Gauss's Remarkable Theorem II

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Please do not hesitate to interrupt me if you have a question.

Review

The Equations and the theorem

- **Codazzi-Mainradi.**

$$\begin{aligned} \mathbb{L}_v - \mathbb{M}_u &= \mathbb{L}\Gamma^1_{12} + \mathbb{M}(\Gamma^2_{12} - \Gamma^1_{11}) - \mathbb{N}\Gamma^2_{11}, \\ \mathbb{M}_v - \mathbb{N}_u &= \mathbb{L}\Gamma^1_{22} + \mathbb{M}(\Gamma^2_{22} - \Gamma^1_{12}) - \mathbb{N}\Gamma^2_{12}. \end{aligned}$$

- **Gauss.**

$$\begin{aligned} \mathbb{E}K &= (\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2, \\ \mathbb{F}K &= (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} \\ &= (\Gamma^2_{12})_v - (\Gamma^2_{22})_u + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11} \\ \mathbb{G}K &= (\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - (\Gamma^1_{12})^2 - \Gamma^2_{12}\Gamma^1_{22}. \end{aligned}$$

Theorem

(Gauss's Theorema Egregium) *The Gaussian curvature of a surface is preseved by local isometries.*

Remarks and Applications of the Gauss and Codazzi-Mainradi Equations

Remarks on the Equations

- For every $\mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{L}, \mathbb{M}, \mathbb{N}$ satisfy these equations, there is a unique surface patch with the two fundamental forms $\mathbb{E}\mathrm{d}u^2 + 2\mathbb{F}\mathrm{d}u\mathrm{d}v + \mathbb{G}\mathrm{d}v^2$ and $\mathbb{L}\mathrm{d}u^2 + 2\mathbb{M}\mathrm{d}u\mathrm{d}v + \mathbb{N}\mathrm{d}v^2$.
- Riemann curvature tensor.

$$R^k_{ijl} = \partial_j \Gamma^k_{il} - \partial_l \Gamma^k_{ij} + \Gamma^k_{jm}\Gamma^m_{il} - \Gamma^k_{lm}\Gamma^m_{ij}$$

where we have used the Einstein summation convention $a^m b_m = a^1 b_1 + a^2 b_2$. Define $R_{ijkl} = g_{im} R^m_{jkl}$. Then $R_{1212} = -(\mathbb{L}\mathbb{N} - \mathbb{M}^2)$.

Examples

Example

Is there a surface with first fundamental form $\mathrm{d}u^2 + \cos^2 u \mathrm{d}v^2$ and $\cos^2 u \mathrm{d}u^2 + \mathrm{d}v^2$?

1. Identify $\mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{L}, \mathbb{M}, \mathbb{N}$;
2. Calculate K and Γ_{ij}^k;
3. Check the Gauss and Codazzi-Mainradi equations.

Surfaces of Constant Gaussian Curvature

Motivation

$\boxed{\text{Gauss's Theorem: Isometric then same } K.}$

Natural question: Same K then isometric?

Answer:

- **No.**
  - Example: $(u, 4v, (u^2 + 4v^2)/2)$ and $(2u, 2v, u^2 + v^2)$;
  - Why are they not isometric?
- **Yes** if K is constant.

$K = 0$

$$\boxed{K = 0 \Rightarrow \text{isometric to plane.}}$$

1. $K = 0 \Rightarrow$ ruled. $\sigma(u, v) = \gamma(u) + v l(u)$.
2. Calculate $\mathbb{E}, \mathbb{F}, \mathbb{G}$.
3. $K = -\|\dot{l}\|^2(1 - (\dot{\gamma} \cdot l)^2) + (\dot{\gamma} \cdot \dot{l})^2$.
4. Two cases.
    4.1 $\dot{l} = 0$. cylinder or plane.
    4.2 $\dot{l} \neq 0$. $\dot{\gamma} \cdot (l \times \dot{l}) = 0$.

$K = constant > 0$

$$\boxed{K = constant > 0 \Rightarrow \text{isometric to a sphere.}}$$

1. Suffices to prove for $K = 1$.
2. Geodesic parametrization: $\mathrm{d}u^2 + \mathbb{G}\mathrm{d}v^2$,
   $\mathbb{G}(0, v) = 1, \mathbb{G}_u(0, v) = 0$.
3. $K = 1$ becomes
   $$-\frac{1}{2}\mathbb{G}_{uu}\mathbb{G} + \frac{1}{4}\mathbb{G}_u^2 = \mathbb{G}^2.$$
4. Set $\mathbb{G} = g^2$,
   $g_{uu} + g = 0 \Rightarrow g(u, v) = A(v)\cos u + B(v)\sin u$.
5. $g(0, v) = 1, g_u(0, v) = 0 \Rightarrow A(v) = 1, B(v) = 0$.
6. First fundamental form $\mathrm{d}u^2 + \cos^2 u\mathrm{d}v^2$.

$K = constant < 0$

$$\boxed{K = constant < 0 \Rightarrow \text{isometric to a pseudosphere.}}$$

- **Pseudosphere.**

  $\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u)),\ f(u) = ae^{u}+be^{-u}.$

- **The proof.** See textbook p. 258.

Compact Surfaces

Theorem

Every connected compact[1] surface whose Gaussian curvature is constant is a sphere.

- Can we drop "connected"?
- Can we drop "compact"?
- "is a sphere" or "is isometry to a sphere"?

[1](in $\mathbb{R}^d$)=closed and bounded

Compact Surfaces: Proof

- Compact surface has at least one p with $K(p) > 0$[2].
- $\kappa_1 > \kappa_2$ **somewhere leads to contradiction.**
  1. $K = \textit{constant} \Rightarrow \kappa_1$ reaches maximum at $p \Rightarrow \kappa_2$ reaches minimum at p;
  2. Around p make $\mathbb{F} = \mathbb{M} = 0$.
  3. Coddazi-Mainradi equations $\Rightarrow$

$$\mathbb{L}_v = \frac{1}{2}\mathbb{E}_v\left(\frac{\mathbb{L}}{\mathbb{E}} + \frac{\mathbb{N}}{\mathbb{G}}\right), \quad \mathbb{N}_u = \frac{1}{2}\mathbb{G}_u\left(\frac{\mathbb{L}}{\mathbb{E}} + \frac{\mathbb{N}}{\mathbb{G}}\right).$$

  4. $\mathbb{E}_v = \mathbb{G}_u = 0$ at p;
  5. $K(p) = -\frac{\mathbb{G}(\kappa_2)_{uu} - \mathbb{E}(\kappa_1)_{vv}}{\mathbb{E}\mathbb{G}(\kappa_1 - \kappa_2)} \leqslant 0$.
  6. $\kappa_1 = \kappa_2$ **everywhere.** Together with $K > 0$ leads to sphere.

[2]Proposition 8.6.1 of textbook

Looking Back and Forward

Summary

$\boxed{\text{Required: }\S 10.1,\ \S 10.2\text{; Optional: }\S 10.3,\ \S 10.4}$

1. Know how to use the equations to check existence of surfaces.
2. Know the isometry results for constant curvature surfaces.

See you next Tuesday!

$\boxed{\text{The Gauss-Bonnet Theorem}}$

- Generalization of $A + B + C = \pi$ for planar triangles.
- Related to the shape of the surface.