Lectures 18: Gauss's Remarkable Theorem II

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

Much of the material in this lecture is optional. On the other hand, it is beneficial to work through the notes as the calculations etc. here can serve as good review of the concepts/formulas we studied in the past two months.

The required textbook sections are §10.1–10.2. The optional sections are §10.3–10.4.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. More discussions on Gauss and Codazzi-Mainradi equations

Remark 1. As Γ_{ij}^k can be calculated using $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$, the Codazzi-Mainradi and Gauss equations can be seen as equations for $\mathbb{L}, \mathbb{M}, \mathbb{N}$ given $\mathbb{E}, \mathbb{F}, \mathbb{G}$.

Remark 2. It can be shown that as long as $\mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{L}, \mathbb{M}, \mathbb{N}$ satisfy the Codazzi-Mainradi and Gauss equations, with Γ_{ij}^k calculated from $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$ as we have seen before, and with $K := \frac{\mathbb{L}\,\mathbb{N} - \mathbb{M}^2}{\mathbb{E}\,\mathbb{G} - \mathbb{F}^2}$, then there is a unique surface patch σ with $\mathbb{E}\, \mathrm{d}u^2 + 2\, \mathbb{F}\, \mathrm{d}u\, \mathrm{d}v + \mathbb{G}\, \mathrm{d}v^2$, $\mathbb{L}\, \mathrm{d}u^2 + 2\, \mathbb{M}\, \mathrm{d}u\, \mathrm{d}v + \mathbb{N}\, \mathrm{d}v^2$ as its first and second fundamental forms, Γ_{ij}^k as its Christoffel symbols, and K as its Gaussian curvature.[1]

Remark 3. (Relation to Riemann Curvature Tensors) We have the following formulas for the Riemann curvature tensor:

$$R_{ijl}^k = \partial_j \Gamma_{il}^k - \partial_l \Gamma_{ij}^k + \Gamma_{j1}^k \Gamma_{il}^1 + \Gamma_{j2}^k \Gamma_{il}^2 - \Gamma_{l1}^k \Gamma_{ij}^1 - \Gamma_{l2}^k \Gamma_{ij}^2. \tag{1}$$

In our context, we have $i, j, k, l = 1, 2$. Thus there are 16 possible R_{ijl}^k's. However it is easy to notice $R_{ijl}^k = -R_{ilj}^k$. Thus we only need to calculate R_{i12}^k.

Setting $u = u^1, v = u^2$, we see that the Gauss equations (2) become

$$\mathbb{F}\, K = R_{112}^1 = -R_{212}^2, \qquad R_{212}^1 = \mathbb{G}\, K, \qquad R_{112}^2 = -\mathbb{E}\, K. \tag{2}$$

Now if we define $R_{ijkl} = g_{im} R_{jkl}^m$, that is

$$\left(\begin{array}{cc} R_{1112} & R_{1212} \\ R_{2112} & R_{2212} \end{array}\right) = \left(\begin{array}{cc} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{array}\right) \left(\begin{array}{cc} R_{112}^1 & R_{212}^1 \\ R_{112}^2 & R_{212}^2 \end{array}\right), \tag{3}$$

We see that

$$\left(\begin{array}{cc} R_{1112} & R_{1212} \\ R_{2112} & R_{2212} \end{array}\right) = (\mathbb{E}\, \mathbb{G} - \mathbb{F}^2) \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right). \tag{4}$$

We see that, among the 16 possible R_{ijkl}, there is exactly one non-trivial value:

$$R_{1212} = -(\mathbb{L}\, \mathbb{N} - \mathbb{M}^2). \tag{5}$$

One now also see that the sectional curvature $K(\sigma_u, \sigma_v) = K$ is exactly the Gaussian curvature of the surface. In general, sectional curvatures are generalizations of Gaussian curvatures.

Example 4. [2]Is there a surface with the first and second fundamental forms $\mathrm{d}u^2 + \cos^2 u\, \mathrm{d}v^2$ and $\cos^2 u\, \mathrm{d}u^2 + \mathrm{d}v^2$?

We see that $\mathbb{E} = 1, \mathbb{F} = 0, \mathbb{G} = \cos^2 u, \mathbb{L} = \cos^2 u, \mathbb{M} = 0, \mathbb{N} = 1$. Now we calculate Γ_{ij}^k. As it is too hard to remember the formulas, we start from the "first principle" (2):

$$\begin{array}{rcl} \sigma_{uu} & = & \Gamma_{11}^1 \sigma_u + \Gamma_{11}^2 \sigma_v + \mathbb{L}\, N, \\ \sigma_{uv} & = & \Gamma_{12}^1 \sigma_u + \Gamma_{12}^2 \sigma_v + \mathbb{M}\, N, \\ \sigma_{vv} & = & \Gamma_{22}^1 \sigma_u + \Gamma_{22}^2 \sigma_v + \mathbb{N}\, N. \end{array} \tag{6}$$

We have

$$\Gamma_{11}^1 = \Gamma_{11}^1 \mathbb{E} + \Gamma_{11}^2 \mathbb{F} = \sigma_{uu} \cdot \sigma_u = \frac{1}{2} \mathbb{E}_u = 0. \tag{7}$$

$$\cos^2 u\, \Gamma_{11}^2 = \Gamma_{11}^1 \mathbb{F} + \Gamma_{11}^2 \mathbb{G} = \sigma_{uu} \cdot \sigma_v = (\sigma_u \cdot \sigma_v)_u - \frac{1}{2} (\sigma_u \cdot \sigma_u)_v = 0 \tag{8}$$

1. Theorem 10.1.3 of the textbook.
2. Exercise 10.1.2 of the textbook.

Thus $\Gamma_{11}^1 = \Gamma_{11}^2 = 0$. Similarly we can calculate

$$\Gamma_{12}^1 = 0, \qquad \Gamma_{12}^2 = -\tan u, \qquad \Gamma_{22}^1 = \cos u \sin u, \qquad \Gamma_{22}^2 = 0. \tag{9}$$

Now we see that

$$\mathbb{M}_v - \mathbb{N}_u = 0 \tag{10}$$

while

$$\mathbb{L}\,\Gamma_{22}^1 + \mathbb{M}\,(\Gamma_{22}^2 - \Gamma_{12}^1) - \mathbb{N}\,\Gamma_{12}^2 = \cos^3 u \sin u + \tan u \neq 0. \tag{11}$$

Thus the second Codazzi-Mainradi equation is not satisfied and consequently there is no surface with the first and second fundamental forms $\mathrm{d}u^2 + \cos^2 u\,\mathrm{d}v^2$ and $\cos^2 u\,\mathrm{d}u^2 + \mathrm{d}v^2$.

Remark 5. One can calculate an explicit formula for K using $\mathbb{E}, \mathbb{F}, \mathbb{G}$ only[3]:

$$K = \frac{\det\begin{pmatrix} -\frac{1}{2}\,\mathbb{E}_{vv} + \mathbb{F}_{uv} - \frac{1}{2}\,\mathbb{G}_{uu} & \frac{1}{2}\,\mathbb{E}_u & \mathbb{F}_u - \frac{1}{2}\,\mathbb{E}_v \\ \mathbb{F}_v - \frac{1}{2}\,\mathbb{G}_u & \mathbb{E} & \mathbb{F} \\ \frac{1}{2}\,\mathbb{G}_v & \mathbb{F} & \mathbb{G} \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}\,\mathbb{E}_v & \frac{1}{2}\,\mathbb{G}_u \\ \frac{1}{2}\,\mathbb{E}_v & \mathbb{E} & \mathbb{F} \\ \frac{1}{2}\,\mathbb{G}_u & \mathbb{F} & \mathbb{G} \end{pmatrix}}{(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)^2}. \tag{12}$$

2. Surfaces of constant Gaussian curvature

By Gauss's Theorem we know that if two surfaces S_1, S_2 are isometric, that is there is a mapping $f: S_1 \mapsto S_2$ such that if $\sigma_1(u, v)$ parametrizes S_1, and $\sigma_2(u, v) := f(\sigma_1(u, v))$, then $\mathbb{E}_1(u, v) = \mathbb{E}_2(u, v)$, $\mathbb{F}_1(u, v) = \mathbb{F}_2(u, v)$, $\mathbb{G}_1(u, v) = \mathbb{G}_2(u, v)$, then $K_1(u, v) = K_2(u, v)$ at every (u, v). The natural question to ask now is

Does the converse hold?

More specifically, we ask, if $\sigma_1(u, v)$ parametrizes S_1 and $\sigma_2(u, v)$ parametrizes S_2, so that $K_1(u, v) = K_2(u, v)$ at every (u, v). Are S_1, S_2 isometric?

- The answer is no.

  Example 6. [4]Let $\sigma_1(u, v) = \left(a\,u, b\,v, \frac{1}{2}\,(a\,u^2 + b\,v^2)\right)$ and $\sigma_2(u, v) = \left(\bar{a}\,u, \bar{b}\,v, \frac{1}{2}\,(\bar{a}\,u^2 + \bar{b}\,v^2)\right)$. Let $|a| \geqslant |b|, |\bar{a}| \geqslant |\bar{b}|$. Then

  a) $K_1(u, v) = K_2(u, v)$ if $a\,b = \bar{a}\,\bar{b}$.

  b) The two surfaces are not locally isometric unless $|a| = |\bar{a}|$ and $|b| = |\bar{b}|$.

  Exercise 1. Prove that the two surfaces are locally isometry when $|a| = |\bar{a}|$ and $|b| = |\bar{b}|$.

  Proof.

  a) For σ_1 we calculate

  $$\mathbb{E} = a^2\,(1 + u^2), \qquad \mathbb{F} = a\,b\,u\,v, \qquad \mathbb{G} = b^2\,(1 + v^2) \tag{13}$$

  and

  $$\mathbb{L} = \frac{a}{\sqrt{1 + u^2 + v^2}}, \qquad \mathbb{M} = 0, \qquad \mathbb{N} = \frac{b}{\sqrt{1 + u^2 + v^2}}. \tag{14}$$

  Thus

  $$K_1 = \frac{1}{a\,b\,(1 + u^2 + v^2)^2}. \tag{15}$$

3. Corollary 10.2.2 of the textbook.

4. Taken from Chen1990.

Similarly we have

$$K_2 = \frac{1}{\bar{a}\,\bar{b}\,(1+u^2+v^2)^2}. \tag{16}$$

The conclusion now follows.

b) Assume the contrary, that is there is an isometry f between the two surfaces. For simplicity of presentation assume $a > b > 0$ and $\bar{a} > \bar{b} > 0$. Other cases are left as exercises.

Now as $K_1(0, 0) = \frac{1}{a\,b} < K_2(u, v)$ whenever $(u, v) \neq (0, 0)$, we must have $f(\sigma_1(0,0)) = \sigma_2(0,0)$.

We write

$$f\left(a\,u, b\,v, \frac{1}{2}\,(a\,u^2 + b\,v^2)\right) = \left(\bar{a}\,U, \bar{b}\,V, \frac{1}{2}\,(\bar{a}\,U^2 + \bar{b}\,V^2)\right) \tag{17}$$

where U, V are functions of u, v with $U(0,0) = V(0,0) = 0$.

As $K_1(u,v) = K_2(U,V)$ we must have $u^2 + v^2 = U^2 + V^2$. Taking derivatives we reach

$$U_u^2 + V_u^2 = 1, \tag{18}$$

$$U_u\,U_v + V_u\,V_v = 0, \tag{19}$$

$$U_v^2 + V_v^2 = 1. \tag{20}$$

At $(0,0)$, we have $\mathbb{E}_1 = a^2, \mathbb{F}_1 = 0, \mathbb{G}_1 = b^2$, and

$$\mathbb{E}_2 = \bar{a}^2\,U_u^2 + \bar{b}^2\,V_u^2, \quad \mathbb{F}_2 = \bar{a}^2\,U_u\,U_v + \bar{b}^2\,V_u\,V_v, \quad \mathbb{G}_2 = \bar{a}^2\,U_v^2 + \bar{b}^2\,V_v^2. \tag{21}$$

There now must hold

$$\bar{a}^2\,U_u^2 + \bar{b}^2\,V_u^2 = a^2, \tag{22}$$

$$\bar{a}^2\,U_u\,U_v + \bar{b}^2\,V_u\,V_v = 0, \tag{23}$$

$$\bar{a}^2\,U_v^2 + \bar{b}^2\,V_v^2 = b^2. \tag{24}$$

As $\bar{a} > \bar{b} > 0$, (19) and (23) now lead to $U_u\,U_v = 0$ and $V_u\,V_v = 0$. Thus there are two cases

i. $U_u = V_v = 0$;

ii. $U_v = V_u = 0$.

The conclusion now easily follows in both cases. □

Remark 7. It is easy to convince ourselves that $a\,b = \bar{a}\,\bar{b}$ are not sufficient for the isometry. Consider $\sigma_\varepsilon(u, v) := \left(\varepsilon\,u, \frac{4}{\varepsilon}\,v, \frac{1}{2}\left(\varepsilon\,u^2 + \frac{4}{\varepsilon}\,v^2\right)\right)$. Now as $\varepsilon \searrow 0$, the surface approaches a “folded plane”. Now consider all the geodesics of length 1 emanating from the the origin on $\sigma_2(u, v)$, their other end points form a circle with radius <1. If there is an isometry between $\sigma_2(u, v)$ and this “folded plane” $\sigma_0(u, v)$, then this circle would be mapped to the union of two semicircles (one on the front side, one on the back) with radius 1 on the “folded plane”. But these two circles must have the same parameter. Contradiction.

- On the other hand, if $K_1(u,v) = K_2(u,v) = \text{constant}$, then the answer is yes. More specifically,
  a) Any surface with $K=0$ everywhere is locally isometric to the plane;
  b) Any surface with $K = \text{constant} > 0$ everywhere is locally isometric to a sphere;
  c) Any surface with $K = \text{constant} < 0$ everywhere is locally isometric to a pseudosphere.

2.1. Surfaces with $K=0$

THEOREM 8. *Let S be a surface covered by one single surface patch. Assume that its Gaussian curvature $K=0$ everywhere. Then S is isometric to an open subset of a plane.*

Proof. We have seen in Lecture 14 (Proposition 11) that such S must be a ruled surface. Thus we can assume the surface patch to be $\sigma(u,v) = \gamma(u) + v\,l(u)$ with $\|l\| = \|\dot{\gamma}\| = 1$. Then we have

$$\sigma_u = \dot{\gamma}(u) + v\,\dot{l}(u), \qquad \sigma_v = l(u). \tag{25}$$

Consequently

$$\mathbb{E} = 1 + 2\,v\,\dot{\gamma}\cdot\dot{l} + v^2\,\|\dot{l}\|^2, \qquad \mathbb{F} = \dot{\gamma}\cdot l, \qquad \mathbb{G} = 1. \tag{26}$$

(12) now gives

$$\begin{aligned} 0 &= \det\begin{pmatrix} -\|\dot{l}\|^2 & \frac{1}{2}\,\mathbb{E}_u & \mathbb{F}_u - \frac{1}{2}\,\mathbb{E}_v \\ 0 & \mathbb{E} & \mathbb{F} \\ 0 & \mathbb{F} & 1 \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}\,\mathbb{E}_v & 0 \\ \frac{1}{2}\,\mathbb{E}_v & \mathbb{E} & \mathbb{F} \\ 0 & \mathbb{F} & 1 \end{pmatrix} \\ &= -\|\dot{l}\|^2\,(\mathbb{E} - \mathbb{F}^2) + \left(\frac{1}{2}\,\mathbb{E}_v\right)^2 \\ &= -\|\dot{l}\|^2\,\left(1 + 2\,v\,(\dot{\gamma}\cdot\dot{l}) + v^2\,\|\dot{l}\|^2 - (\dot{\gamma}\cdot l)^2\right) + \left(\dot{\gamma}\cdot\dot{l} + v\,\|\dot{l}\|^2\right)^2 \\ &= -\|\dot{l}\|^2\,(1 - (\dot{\gamma}\cdot l)^2) + (\dot{\gamma}\cdot\dot{l})^2. \end{aligned} \tag{27}$$

We discuss two cases.

i. $\dot{l} = 0$. In this case σ is an open subset of a generalized cylinder and is isometric to an open subset of a plane.

ii. $\dot{l} \neq 0$. Let $\hat{\dot{l}} := \frac{\dot{l}}{\|\dot{l}\|}$, from (27) we have

$$1 - (\dot{\gamma}\cdot l)^2 - \left(\dot{\gamma}\cdot\hat{\dot{l}}\right)^2 = 0. \tag{28}$$

As $\dot{\gamma}$ is a unit vector and so are $l, \hat{\dot{l}}$ and furthermore $l \perp \hat{\dot{l}}$, we see that $\dot{\gamma}\cdot\left(l\times\hat{\dot{l}}\right) = 0$ and therefore $\dot{\gamma}\cdot\left(l\times\dot{l}\right) = 0$. From §3 of Lecture 10 we see that this implies S is developable and consequently is isometric to an open subset of a plane. $\square$

2.2. Surfaces with $K>0$ constant

THEOREM 9. *Let S be a surface covered by one single surface patch. Assume that its Gaussian curvature K is a positive constant. Then S is isometric to an open subset of a sphere.*

Proof. Clearly it suffices to consider the case $K = 1$.

Exercise 2. Rigorously justify this.

We first re-parametrize the surface so that the fundamental forms are simple.

Geodesic coordinates

PROPOSITION 10. (PROPOSITION 9.5.1 OF THE TEXTBOOK) *Let $p_0 \in S$. Then there is a neighborhood of p that can be parametrized so that the first fundamental form is $\mathrm{d}u^2 + \mathbb{G}(u, v)\,\mathrm{d}v^2$ where $\mathbb{G}(u, v)$ is a smooth function satisfying ($p_0 = \sigma(0, 0)$)*

$$\mathbb{G}(0, v) = 1, \qquad \mathbb{G}_u(0, v) = 0. \tag{29}$$

Proof. Let $\gamma(v)$ be a geodesic passing p with v the arc length parameter. At each point on γ, let $\tilde{\gamma}^v(u)$ be the geodesic passing that point and is perpendicular to γ. Further assume that u is also the arc length parameter.

Exercise 3. Explain why there is a neighborhood of p that is fully covered by these $\tilde{\gamma}^v(u)$'s.

Exercise 4. Explain why each point in this neighborhood belongs to exactly one $\tilde{\gamma}^v(u)$.

Thus we obtain a surface patch $\sigma(u, v) = \tilde{\gamma}^v(u)$. As for each v, $\tilde{\gamma}^v(u)$ is arc length parametrized, there holds $\|\sigma_u\| = 1 \Longrightarrow \mathbb{E} = 1$. To show that $\mathbb{F} = 0$, we notice that by construction

$$\sigma_u(0, v_0) \cdot \sigma_v(0, v_0) = 0. \tag{30}$$

On the other hand, we parametrize $\tilde{\gamma}^{v_0}(u)$ by $u(t) = t, v(t) = v_0$. Thus we have $\dot{u} = 1, \dot{v} = 0$. As $\mathbb{E} = 1$, we see that this is arc length parametrization. The geodesic equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{E}\,\dot{u} + \mathbb{F}\,\dot{v}) = \frac{1}{2}(\dot{u}, \dot{v})\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}_u \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \tag{31}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{F}\,\dot{u} + \mathbb{G}\,\dot{v}) = \frac{1}{2}(\dot{u}, \dot{v})\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}_v \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \tag{32}$$

now become

$$0 = 0 \text{ and } \dot{\mathbb{F}} = 0. \tag{33}$$

Thus $\mathbb{F} = 0$ along $\tilde{\gamma}^{v_0}$, and consequently for all u, v as v_0 is arbitrary.

Exercise 5. Explain, using the intuition "geodesics are shortest paths", why $\sigma(u_0, v)$ has to be perpendicular to $\sigma(u, v_0)$.

Finally, $\sigma_v(0, v) = \frac{\mathrm{d}\gamma}{\mathrm{d}v}$ so $\mathbb{G}(0, v) = 1$. Then (31) applied to the geodesic $\gamma(v) = \sigma(0, v)$ gives $\mathbb{G}_u(0, v) = 0$. □

(12) now gives

$$-\frac{1}{2}\,\mathbb{G}_{uu}\,\mathbb{G} + \frac{1}{4}\,\mathbb{G}_u^2 = \mathbb{G}^2. \tag{34}$$

Setting $\mathbb{G} = g^2$ we reach

$$g_{uu} + g = 0 \Longrightarrow g(u,v) = A(v)\cos u + B(v)\sin u. \tag{35}$$

Thanks to (29) there holds $g(0,v) = 1, g_u(0,v) = 0$ which give $A(v) = 1, B(v) = 0$. Consequently we have the first fundamental form to be

$$\mathrm{d}u^2 + \cos^2 u\,\mathrm{d}v^2 \tag{36}$$

which is exactly the first fundamental form of $\mathbb{S}^2$ in spherical coordinates. □

2.3. Surfaces with $K < 0$ constant

THEOREM 11. *Let S be a surface covered by one single surface patch. Assume that its Gaussian curvature K is a negative constant. Then S is isometric to an open subset of a pseudosphere.*

The pseudosphere

Consider a surface of revolution $\sigma(u,v) = (f(u)\cos v,\, f(u)\sin v,\, g(u))$ where $f > 0$ and $(f')^2 + (g')^2 = 1$.

Exercise 6. Show that $K = -\frac{f''}{f}$.

When $K = -1$, we have $f'' - f = 0 \Longrightarrow f(u) = a\,e^u + b\,e^{-u}$. Such a surface is called a "pseudosphere". For example when $a = 1, b = 0$ we have

$$g(u) = \int \sqrt{1 - e^{2u}}\,\mathrm{d}u = \sqrt{1 - e^{2u}} - \ln\left(e^{-u} + \sqrt{e^{-2u} - 1}\right). \tag{37}$$

Exercise 7. Draw a sketch of the pseudosphere in this case.

Proof. (OF THEOREM 11) Same idea as the proof of Theorem 9. See p.258 of the textbook. □

2.4. Compact surfaces

THEOREM 12. [5]*Every connected compact surface whose Gaussian curvature is constant is a sphere.*

Exercise 8. Every compact surface has a point where $K \geqslant 0$. (Hint: Consider $p \in S$ that is furthest away from the origin)

Proof. Consider the function $J = (\kappa_1 - \kappa_2)^2$. Let p be where J reaches maximum–this is possible as J is continuous. If this maximum is 0 then we have $\kappa_1 = \kappa_2$ and S must be part of a sphere. If not, thanks to $\kappa_1\kappa_2 = K$ being constant, there must hold that κ_1 is at local maximum and κ_2 at local minimum at p. We reach contradiction using the following lemma.

5. Theorem 10.3.4 of the textbook.

LEMMA 13. [6]*Let* $\sigma : U \mapsto \mathbb{R}^3$ *be a surface patch containing a point* p *that is not an umbilic*[7]. *Let* $\kappa_1 \geqslant \kappa_2$ *be the principal curvatures of* σ *and suppose that* κ_1 *has a local maximum at* p *and* κ_2 *has a local minimum at* p*. Then* $K(p) \leqslant 0$. □

Proof. (OF LEMMA 13) Taking a small neighborhood of p such that $\kappa_1 > \kappa_2$ in it. Take the coordinate system along the principal vectors so that the two fundamental forms are

$$\mathbb{E}\, \mathrm{d}u^2 + \mathbb{G}\, \mathrm{d}v^2, \qquad \mathbb{L}\, \mathrm{d}u^2 + \mathbb{N}\, \mathrm{d}v^2. \tag{38}$$

Thus we have $\kappa_1 = \frac{\mathbb{L}}{\mathbb{E}}$ and $\kappa_2 = \frac{\mathbb{N}}{\mathbb{G}}$.

Exercise 9. What about $\kappa_1 > \kappa_2$?

The Codazzi-Mainradi equations now become

$$\mathbb{L}_v = \frac{1}{2}\, \mathbb{E}_v \left(\frac{\mathbb{L}}{\mathbb{E}} + \frac{\mathbb{N}}{\mathbb{G}} \right), \qquad \mathbb{N}_u = \frac{1}{2}\, \mathbb{G}_u \left(\frac{\mathbb{L}}{\mathbb{E}} + \frac{\mathbb{N}}{\mathbb{G}} \right) \tag{39}$$

and consequently

$$\mathbb{E}_v = \left(\frac{2\, \mathbb{E}}{\kappa_2 - \kappa_1} \right) (\kappa_1)_v, \qquad \mathbb{G}_u = \left(\frac{2\, \mathbb{G}}{\kappa_1 - \kappa_2} \right) (\kappa_2)_u. \tag{40}$$

At p we have $(\kappa_1)_v = (\kappa_2)_u = 0 \Longrightarrow \mathbb{E}_v = \mathbb{G}_u = 0$ which leads to

$$\begin{aligned} K(p) &= -\frac{1}{2\sqrt{\mathbb{E}\,\mathbb{G}}} \left(\frac{\partial}{\partial u} \left(\frac{\mathbb{G}_u}{\sqrt{\mathbb{E}\,\mathbb{G}}} \right) + \frac{\partial}{\partial v} \left(\frac{\mathbb{E}_v}{\sqrt{\mathbb{E}\,\mathbb{G}}} \right) \right) \\ &= -\frac{\mathbb{G}_{uu} + \mathbb{E}_{vv}}{2\,\mathbb{E}\,\mathbb{G}} \\ &= -\frac{\mathbb{G}\,(\kappa_2)_{uu} - \mathbb{E}\,(\kappa_1)_{vv}}{\mathbb{E}\,\mathbb{G}\,(\kappa_1 - \kappa_2)} \leqslant 0, \end{aligned} \tag{41}$$

where the last inequality comes from κ_1 taking a local maximum and κ_2 taking a local minimum at p . □

Remark 14. Similar to the proof of Theorem 12 we can show that a compact surface with $K > 0$ everywhere and constant H is a sphere.

6. Lemma 10.3.5 of the textbook.

7. that is $\kappa_1 \neq \kappa_2$.