Math 348 Differential Geometry of Curves and Surfaces

Lecture 17: Gauss's Remarkable Theorem I

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Please do not hesitate to interrupt me if you have a question.

Review of Midterm 2

Overview

High:15(100%);Low:7(47%);Average:11.37(76%);Median:12(80%)

1. **Question 1.** Average: 3.96; Median: 4.
2. **Question 2.** Average: 4.44; Median: 5.
3. **Question 3.** Average: 2.89; Median: 3.
4. **Question 4.** Average: 0.07; Median: 0.

Question 1

$$\boxed{\sigma(u,v) = (u, v, u^2 - v^2).\ \cos\theta \text{ between } u = 2 \text{ and } u = 2v.}$$

Major mistakes.

- Incorrect parametrization of the curves. e.g. $u = 2v \to \sigma(t, 2t)$.
- Where to evaluate $\mathbb{E}, \mathbb{F}, \mathbb{G}$?
- Incorrect v_1, v_2, w_1, w_2.
- $\langle v, v\rangle = \mathbb{E}v_1^2 + \mathbb{F}v_1v_2 + \mathbb{G}v_2^2$.
- Calculation mistakes.
- Incorrect $\mathbb{E}, \mathbb{F}, \mathbb{G}$. In particular, there must hold $\mathbb{E}, \mathbb{G} > 0$ and $\mathbb{E}\mathbb{G} > \mathbb{F}^2$.

Question 2

$$\boxed{\sigma(u,v) = (u, v, u^2 - v^2). \text{ Curvatures at } p = (1,0,1).}$$

Major mistakes.

- $p = \sigma(1,1)$.
- $N = \frac{(-2,0,1)}{\sqrt{2}}$.
- $\det\left[\begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix} - \kappa \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}\right] = 0$.
- How to solve $\det\begin{pmatrix} \frac{2}{\sqrt{5}} - 5\kappa & 0 \\ 0 & -\frac{2}{\sqrt{5}} - \kappa \end{pmatrix} = 0$?

Question 3

$\boxed{\sigma(u,v) = (u, v, u^2 - v^2).\ \text{Calculate } \Gamma_{ij}^k.\ v = 0 \text{ pre-geodesic?}}$

Major mistakes.

- a) Γ_{ij}^k.
  - Calculation mistakes.
  - $\Gamma_{11}^1 = 4/5, \ldots$; "From last page ... $\mathbb{E} = 5, \ldots$".
- b).
  - $v(t) = 0, u(t) = t$.

Question 4

$$\boxed{S_1 \cap S_2 = C.\ \ S_1, S_2 \text{ not tangent. } C \text{ is pre-geodesic in } S_1.}$$

Major mistakes.

Gauss and Codazzi-Mainradi Equations

Motivation

$\boxed{S\text{: surface patch. Two fundamental forms. Six Christoffel symbols.}}$

- Twelve functions of (u, v).
- Γ_{ij}^{k} are determined by $\mathbb{E}, \mathbb{F}, \mathbb{G}$.
- Any relations between the two fundamental forms?

The Equations

- **Codazzi-Mainradi.**

$$\begin{aligned} \mathbb{L}_v - \mathbb{M}_u &= \mathbb{L}\Gamma^1_{12} + \mathbb{M}(\Gamma^2_{12} - \Gamma^1_{11}) - \mathbb{N}\Gamma^2_{11}, \\ \mathbb{M}_v - \mathbb{N}_u &= \mathbb{L}\Gamma^1_{22} + \mathbb{M}(\Gamma^2_{22} - \Gamma^1_{12}) - \mathbb{N}\Gamma^2_{12}. \end{aligned}$$

- **Gauss.**

$$\begin{aligned} \mathbb{E}K &= (\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2, \\ \mathbb{F}K &= (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} \\ &= (\Gamma^2_{12})_v - (\Gamma^2_{22})_u + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11} \\ \mathbb{G}K &= (\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - (\Gamma^1_{12})^2 - \Gamma^2_{12}\Gamma^1_{22}. \end{aligned}$$

Proof

The equations follow from

1. $\sigma_{uuv} = \sigma_{uvu}$ and $\sigma_{uvv} = \sigma_{vvu}$;
2. $-N_u = a_{11}\sigma_u + a_{12}\sigma_v$ and $-N_v = a_{21}\sigma_u + a_{22}\sigma_v$, with $\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{E} & \mathbb{F} \\ \mathbb{F} & \mathbb{G} \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{L} & \mathbb{M} \\ \mathbb{M} & \mathbb{N} \end{pmatrix}$
3. Serious calculation.

Gauss's Theorema Egregium

Theorem

(Gauss's Theorema Egregium) *The Gaussian curvature of a surface is presefved by local isometries.*

Proof.

Follows immediately from the Gauss equations. □

Looking Back and Forward

Summary

$\boxed{\text{Required: }\S 10.1,\ \S 10.2\text{; Optional: }\S 10.3,\ \S 10.4}$

1. Know the existence of the equations.

See you Thursday!

$\boxed{\text{Applications of the Gauss and Codazzi-Mainradi equations.}}$

- Existence of surfaces;
- Surfaces with constant Gaussian curvature.