Lectures 17: Gauss's Remarkable Theorem I

Disclaimer. As we have a textbook, this lecture note is for guidance and supplement only. It should not be relied on when preparing for exams.

Much of the material in this lecture is optional. On the other hand, it is beneficial to work through the notes as the calculations etc. here can serve as good review of the concepts/formulas we studied in the past two months.

The required textbook sections are §10.1–10.2. The optional sections are §10.3–10.4.

I try my best to make the examples in this note different from examples in the textbook. Please read the textbook carefully and try your hands on the exercises. During this please don't hesitate to contact me if you have any questions.

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1. Motivation

We have seen that given a surface, one can calculate its first and second fundamental forms

$$\mathbb{E}\,\mathrm{d}u^2 + 2\,\mathbb{F}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{G}\,\mathrm{d}v^2 \text{ and } \mathbb{L}\,\mathrm{d}u^2 + 2\,\mathbb{M}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{N}\,\mathrm{d}v^2, \tag{1}$$

as well as the Christoffel symbols Γ_{ij}^k, defined through

$$\begin{aligned} \sigma_{uu} &= \Gamma_{11}^1 \sigma_u + \Gamma_{11}^2 \sigma_v + \mathbb{L}\,N, \\ \sigma_{uv} &= \Gamma_{12}^1 \sigma_u + \Gamma_{12}^2 \sigma_v + \mathbb{M}\,N, \\ \sigma_{vv} &= \Gamma_{22}^1 \sigma_u + \Gamma_{22}^2 \sigma_v + \mathbb{N}\,N. \end{aligned} \tag{2}$$

We have also seen that the Christoffel symbols are not independent quantities and can be calculated from $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$. Now the question is, are $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$ and $\mathbb{L}$, $\mathbb{M}$, $\mathbb{N}$ independent? In other words, given six functions $\mathbb{E}(u,v), \ldots, \mathbb{N}(u,v)$, is there always a surface S having (1) as its first and second fundamental forms?

For example, is there a surface with first and second fundamental forms $\mathrm{d}u^2 + \cos^2 u\,\mathrm{d}v^2$ and $\cos^2 u\,\mathrm{d}u^2 + \mathrm{d}v^2$? To answer this, we need to first understand whether $\mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{L}, \mathbb{M}, \mathbb{N}$ are related.

2. Codazzi-Mainradi equations and Gauss equations

THEOREM 1. *Let S be a surface and let $\mathbb{E}\,\mathrm{d}u^2 + 2\,\mathbb{F}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{G}\,\mathrm{d}v^2$, $\mathbb{L}\,\mathrm{d}u^2 + 2\,\mathbb{M}\,\mathrm{d}u\,\mathrm{d}v + \mathbb{N}\,\mathrm{d}v^2$, and Γ_{ij}^k be its first, second fundamental forms, and Christoffel symbols. Then there hold*

- *the Codazzi-Mainradi equations*

$$\begin{aligned} \mathbb{L}_v - \mathbb{M}_u &= \mathbb{L}\,\Gamma_{12}^1 + \mathbb{M}\,(\Gamma_{12}^2 - \Gamma_{11}^1) - \mathbb{N}\,\Gamma_{11}^2 \\ \mathbb{M}_v - \mathbb{N}_u &= \mathbb{L}\,\Gamma_{22}^1 + \mathbb{M}\,(\Gamma_{22}^2 - \Gamma_{12}^1) - \mathbb{N}\,\Gamma_{12}^2 \end{aligned}, \tag{3}$$

- *and the Gauss equations*

$$\begin{aligned} \mathbb{E}\,K &= (\Gamma_{11}^2)_v - (\Gamma_{12}^2)_u + \Gamma_{11}^1 \Gamma_{12}^2 + \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{12}^1 \Gamma_{11}^2 - (\Gamma_{12}^2)^2, \\ \mathbb{F}\,K &= (\Gamma_{12}^1)_u - (\Gamma_{11}^1)_v + \Gamma_{12}^2 \Gamma_{12}^1 - \Gamma_{11}^2 \Gamma_{22}^1, \\ &= (\Gamma_{12}^2)_v - (\Gamma_{22}^2)_u + \Gamma_{12}^1 \Gamma_{12}^2 - \Gamma_{22}^1 \Gamma_{11}^2, \\ \mathbb{G}\,K &= (\Gamma_{22}^1)_u - (\Gamma_{12}^1)_v + \Gamma_{22}^1 \Gamma_{11}^1 + \Gamma_{22}^2 \Gamma_{12}^1 - (\Gamma_{12}^1)^2 - \Gamma_{12}^2 \Gamma_{22}^1. \end{aligned} \tag{4}$$

Proof. By (2) we have

$$(\Gamma_{11}^1 \sigma_u + \Gamma_{11}^2 \sigma_v + \mathbb{L}\,N)_v = \sigma_{uuv} = \sigma_{uvu} = (\Gamma_{12}^1 \sigma_u + \Gamma_{12}^2 \sigma_v + \mathbb{M}\,N)_u \tag{5}$$

and

$$(\Gamma_{12}^1 \sigma_u + \Gamma_{12}^2 \sigma_v + \mathbb{M}\,N)_v = \sigma_{uvv} = \sigma_{vvu} = (\Gamma_{22}^1 \sigma_u + \Gamma_{22}^2 \sigma_v + \mathbb{N}\,N)_u. \tag{6}$$

Now calculate

$$\begin{aligned}(\Gamma_{11}^{1}\sigma_u+\Gamma_{11}^{2}\sigma_v+\mathbb{L}\,N)_v &= (\Gamma_{11}^{1})_v\sigma_u+\Gamma_{11}^{1}\sigma_{uv}+(\Gamma_{11}^{2})_v\sigma_v+\Gamma_{11}^{2}\sigma_{vv}+\mathbb{L}_v N+\mathbb{L}\,N_v\\ &= (\Gamma_{11}^{1})_v\sigma_u+(\Gamma_{11}^{2})_v\sigma_v+\mathbb{L}_v N\\ &\quad+\Gamma_{11}^{1}(\Gamma_{12}^{1}\sigma_u+\Gamma_{12}^{2}\sigma_v+\mathbb{M}\,N)\\ &\quad+\Gamma_{11}^{2}(\Gamma_{22}^{1}\sigma_u+\Gamma_{22}^{2}\sigma_v+\mathbb{N}\,N)\\ &\quad+\mathbb{L}\,(-a_{21}\sigma_u-a_{22}\sigma_v)\\ &= [(\Gamma_{11}^{1})_v+\Gamma_{11}^{1}\Gamma_{12}^{1}+\Gamma_{11}^{2}\Gamma_{22}^{1}-a_{21}\mathbb{L}]\,\sigma_u\\ &\quad+[(\Gamma_{11}^{2})_v+\Gamma_{11}^{1}\Gamma_{12}^{2}+\Gamma_{11}^{2}\Gamma_{22}^{2}-a_{22}\mathbb{L}]\,\sigma_v\\ &\quad+[\mathbb{L}_v+\Gamma_{11}^{1}\mathbb{M}+\Gamma_{11}^{2}\mathbb{N}]\,N. \end{aligned}\tag{7}$$

Here recall that $\begin{pmatrix} a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}=\begin{pmatrix} \mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}^{-1}\begin{pmatrix} \mathbb{L} & \mathbb{M}\\ \mathbb{M} & \mathbb{N}\end{pmatrix}$.

Similarly we calculate

$$\begin{aligned}(\Gamma_{12}^{1}\sigma_u+\Gamma_{12}^{2}\sigma_v+\mathbb{M}\,N)_u &= (\Gamma_{12}^{1})_u\sigma_u+\Gamma_{12}^{1}\sigma_{uu}+(\Gamma_{12}^{2})_u\sigma_v+\Gamma_{12}^{2}\sigma_{uv}+\mathbb{M}_u N+\mathbb{M}\,N_u\\ &= (\Gamma_{12}^{1})_u\sigma_u+(\Gamma_{12}^{2})_u\sigma_v+\mathbb{M}_u N\\ &\quad+\Gamma_{12}^{1}(\Gamma_{11}^{1}\sigma_u+\Gamma_{11}^{2}\sigma_v+\mathbb{L}\,N)\\ &\quad+\Gamma_{12}^{2}(\Gamma_{12}^{1}\sigma_u+\Gamma_{12}^{2}\sigma_v+\mathbb{M}\,N)\\ &\quad+\mathbb{M}\,(-a_{11}\sigma_u-a_{12}\sigma_v)\\ &= [(\Gamma_{12}^{1})_u+\Gamma_{12}^{1}\Gamma_{11}^{1}+\Gamma_{12}^{2}\Gamma_{12}^{1}-a_{11}\mathbb{M}]\,\sigma_u\\ &\quad+[(\Gamma_{12}^{2})_u+\Gamma_{12}^{1}\Gamma_{11}^{2}+\Gamma_{12}^{2}\Gamma_{12}^{2}-a_{12}\mathbb{M}]\,\sigma_v\\ &\quad+[\mathbb{M}_u+\Gamma_{12}^{1}\mathbb{L}+\Gamma_{12}^{2}\mathbb{M}]\,N. \end{aligned}\tag{8}$$

As $\{\sigma_u,\sigma_v,N\}$ form a basis of $\mathbb{R}^3$, there must hold

$$(\Gamma_{11}^{1})_v+\Gamma_{11}^{1}\Gamma_{12}^{1}+\Gamma_{11}^{2}\Gamma_{22}^{1}-a_{21}\mathbb{L} = (\Gamma_{12}^{1})_u+\Gamma_{12}^{1}\Gamma_{11}^{1}+\Gamma_{12}^{2}\Gamma_{12}^{1}-a_{11}\mathbb{M},\tag{9}$$

$$(\Gamma_{11}^{2})_v+\Gamma_{11}^{1}\Gamma_{12}^{2}+\Gamma_{11}^{2}\Gamma_{22}^{2}-a_{22}\mathbb{L} = (\Gamma_{12}^{2})_u+\Gamma_{12}^{1}\Gamma_{11}^{2}+\Gamma_{12}^{2}\Gamma_{12}^{2}-a_{12}\mathbb{M},\tag{10}$$

$$\mathbb{L}_v+\Gamma_{11}^{1}\mathbb{M}+\Gamma_{11}^{2}\mathbb{N} = \mathbb{M}_u+\Gamma_{12}^{1}\mathbb{L}+\Gamma_{12}^{2}\mathbb{M}.\tag{11}$$

We see that (11) immediately gives the first Codazzi-Mainradi equation. On the other hand, (9) yields

$$a_{11}\mathbb{M}-a_{21}\mathbb{L}=(\Gamma_{12}^{1})_u-(\Gamma_{11}^{1})_v+\Gamma_{12}^{2}\Gamma_{12}^{1}-\Gamma_{11}^{2}\Gamma_{22}^{1},\tag{12}$$

which becomes the second Gauss equation after we notice the following.

$$\begin{aligned} a_{11}\mathbb{M}-a_{21}\mathbb{L} &= -\left[\begin{pmatrix} a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}\begin{pmatrix} \mathbb{N} & -\mathbb{M}\\ -\mathbb{M} & \mathbb{L}\end{pmatrix}\right]_{(12)}\\ &= -\left[\begin{pmatrix} \mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}^{-1}\begin{pmatrix} \mathbb{L} & \mathbb{M}\\ \mathbb{M} & \mathbb{N}\end{pmatrix}\begin{pmatrix} \mathbb{N} & -\mathbb{M}\\ -\mathbb{M} & \mathbb{L}\end{pmatrix}\right]_{(12)}\\ &= -(\mathbb{L}\,\mathbb{N}-\mathbb{M}^2)\left[\begin{pmatrix} \mathbb{E} & \mathbb{F}\\ \mathbb{F} & \mathbb{G}\end{pmatrix}^{-1}\right]_{(12)}\\ &= -(\mathbb{L}\,\mathbb{N}-\mathbb{M}^2)\,(\mathbb{E}\,\mathbb{G}-\mathbb{F}^2)^{-1}\begin{pmatrix} \mathbb{G} & -\mathbb{F}\\ -\mathbb{F} & \mathbb{E}\end{pmatrix}_{(12)}\\ &= \frac{\mathbb{L}\,\mathbb{N}-\mathbb{M}^2}{\mathbb{E}\,\mathbb{G}-\mathbb{F}^2}\,\mathbb{F}=\mathbb{F}\,K. \end{aligned}\tag{13}$$

Here we use $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{(ij)}$ to denote the (i,j) component of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Similar calculations confirm the rest of the equations. □

3. Gauss's remarkable theorem

THEOREM 2. (GAUSS'S THEOREMA EGREGIUM) *The Gaussian curvature of a surface is preserved by local isometries.*

Proof. This follows immediately from the Gauss equations together with the fact that if $f\colon S \mapsto \tilde{S}$ is a local isometry, then the first fundamental forms for $\sigma(u, v)$ and $\tilde{\sigma}(u, v) := f(\sigma(u,v))$ are identical, and consequently the two surfaces have the same $\mathbb{E}, \mathbb{F}, \mathbb{G}, \Gamma_{ij}^k$. □

Remark 3. One can calculate an explicit formula for K using $\mathbb{E}, \mathbb{F}, \mathbb{G}$ only[1]:

$$K = \frac{\det\begin{pmatrix} -\frac{1}{2}\mathbb{E}_{vv} + \mathbb{F}_{uv} - \frac{1}{2}\mathbb{G}_{uu} & \frac{1}{2}\mathbb{E}_u & \mathbb{F}_u - \frac{1}{2}\mathbb{E}_v \\ \mathbb{F}_v - \frac{1}{2}\mathbb{G}_u & \mathbb{E} & \mathbb{F} \\ \frac{1}{2}\mathbb{G}_v & \mathbb{F} & \mathbb{G} \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}\mathbb{E}_v & \frac{1}{2}\mathbb{G}_u \\ \frac{1}{2}\mathbb{E}_v & \mathbb{E} & \mathbb{F} \\ \frac{1}{2}\mathbb{G}_u & \mathbb{F} & \mathbb{G} \end{pmatrix}}{(\mathbb{E}\,\mathbb{G} - \mathbb{F}^2)^2}. \tag{14}$$

1. Corollary 10.2.2 of the textbook.