Math 348 Differential Geometry of Curves and Surfaces

Lecture 16 Geodesics

Xinwei Yu

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CAB 527, xinwei2@ualberta.ca
Department of Mathematical & Statistical Sciences
University of Alberta

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Please do not hesitate to interrupt me if you have a question.

Review

Parallel Transport

$$\boxed{\text{Covariant derivative: } \nabla_\gamma w = \dot{w} - (\dot{w}\cdot N_S)N_S}$$

- **Parallel transport.** w: A tangent vector field defined along a curve $\gamma(t)$.
  - w is parallel along γ: $\nabla_\gamma w = 0$;
  - $w = \alpha\sigma_u + \beta\sigma_v$ then w is parallel along $\gamma \Leftrightarrow$

$$\begin{aligned} \dot{\alpha} + (\Gamma^1_{11}\dot{u} + \Gamma^1_{12}\dot{v})\alpha + (\Gamma^1_{12}\dot{u} + \Gamma^1_{22}\dot{v})\beta &= 0 \quad (1) \\ \dot{\beta} + (\Gamma^2_{11}\dot{u} + \Gamma^2_{12}\dot{v})\alpha + (\Gamma^2_{12}\dot{u} + \Gamma^2_{22}\dot{v})\beta &= 0. \quad (2) \end{aligned}$$

- Christoffel symbols.

$$\begin{aligned} \sigma_{uu} &= \Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_v + \mathbb{L}N \quad (3) \\ \sigma_{uv} &= \Gamma^1_{12}\sigma_u + \Gamma^2_{12}\sigma_v + \mathbb{M}N \quad (4) \\ \sigma_{vv} &= \Gamma^1_{22}\sigma_u + \Gamma^2_{22}\sigma_v + \mathbb{N}N \quad (5) \end{aligned}$$

Calculation of Γ_{ij}^k: An Example

$$\boxed{\sigma(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)}$$

1. Preparation. Calculate

$$\sigma_u, \sigma_v, N, \sigma_{uu}, \sigma_{uv}, \sigma_{vv}.$$

2. Solve for Γ_{11}^k.

$$\sigma_{uu} = \Gamma_{11}^1 \sigma_u + \Gamma_{11}^2 \sigma_v + \mathbb{L} N$$

3. Solve for Γ_{12}^k.

$$\sigma_{uv} = \Gamma_{12}^1 \sigma_u + \Gamma_{12}^2 \sigma_v + \mathbb{M} N$$

4. Solve for Γ_{22}^k.

$$\sigma_{vv} = \Gamma_{22}^1 \sigma_u + \Gamma_{22}^2 \sigma_v + \mathbb{N} N.$$

Geodesics

"Straight" lines on a surface

Meaning of "straight"?

1. $\kappa = |\kappa_n|$ along the curve;
2. $\kappa_g = 0$ along the curve;
3. $\nabla_\gamma T = 0$ along the curve.
4. Shortest path connecting two points.

All four characterizations roughly equivalent.

$$\boxed{\textbf{Definition.}\ \gamma(t) \text{ is a geodesic: } \nabla_\gamma \dot{\gamma} = 0}$$

- γ is a geodesic $\Rightarrow \|\dot{\gamma}\|$ is constant.

Geodesic equations

$$\boxed{\gamma(t) = \sigma(u(t), v(t)).}$$

If $\|\dot{\gamma}\|$ is constant, then γ is a geodesic $\Leftrightarrow$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{E}\dot{u} + \mathbb{F}\dot{v}) &= \frac{1}{2}(\mathbb{E}_u\dot{u}^2 + 2\mathbb{F}_u\dot{u}\dot{v} + \mathbb{G}_u\dot{v}^2), \qquad (6) \\ \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{F}\dot{u} + \mathbb{G}\dot{v}) &= \frac{1}{2}(\mathbb{E}_v\dot{u}^2 + 2\mathbb{F}_v\dot{u}\dot{v} + \mathbb{G}_v\dot{v}^2). \qquad (7) \end{aligned}$$

- Called **geodesic equations**.
- Consequence of $\kappa = |\kappa_n|$.
- Among the three equations: two geodesic equations and $\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\gamma}\| = 0$, any two imply the third.

Alternative formulations

$\nabla_\gamma \dot{\gamma} = 0 \Leftrightarrow$ parallel transport equation

$$\begin{aligned} \dot{\alpha} + (\Gamma^1_{11}\dot{u} + \Gamma^1_{12}\dot{v})\alpha + (\Gamma^1_{12}\dot{u} + \Gamma^1_{22}\dot{v})\beta &= 0 \qquad (8) \\ \dot{\beta} + (\Gamma^2_{11}\dot{u} + \Gamma^2_{12}\dot{v})\alpha + (\Gamma^2_{12}\dot{u} + \Gamma^2_{22}\dot{v})\beta &= 0. \qquad (9) \end{aligned}$$

where $\alpha = \dot{u}, \beta = \dot{v}$, thanks to $\dot{\gamma} = \dot{u}\sigma_u + \dot{v}\sigma_v$,

$$\begin{aligned} \ddot{u} + \Gamma^1_{11}\dot{u}^2 + 2\Gamma^1_{12}\dot{u}\dot{v} + \Gamma^1_{22}\dot{v}^2 &= 0 \qquad (10) \\ \ddot{v} + \Gamma^2_{11}\dot{u}^2 + 2\Gamma^2_{12}\dot{u}\dot{v} + \Gamma^2_{22}\dot{v}^2 &= 0. \qquad (11) \end{aligned}$$

- The geodesic equations are easier to remember in matrix form.

$\|\dot{\gamma}\|$ does not need to be constant in the parallel transport equations. What has changed here?

Application to Surfaces of Revolution

Clairaut's Theorem

$$\boxed{\sigma(u,v) = (f(u)\cos v, f(u)\sin v, u).}$$

1. Geodesic equations.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}((1+f'(u)^2)\dot{u}) &= f'(u)f''(u)\dot{u}^2 + f(u)f'(u)\dot{v}^2, \\ \frac{\mathrm{d}}{\mathrm{d}s}(f(u)^2\dot{v}) &= 0. \end{aligned}$$

2. Assume $(1+f'(u)^2)\dot{u}^2 + f(u)^2\dot{v}^2 = \|\dot{\gamma}\|^2 = 1$. Then

$$\cos\angle(\sigma_u, \dot{\gamma}) = \sqrt{1+f'(u)^2}\,\dot{u}.$$

3. Consequently,

$$f(u)\sin\angle(\sigma_u, \dot{\gamma}) = Constant.$$

4. Clairaut's Theorem: Geodesic $\Leftrightarrow f(u)\sin\angle(\sigma_u, \dot{\gamma}) = C$.

Geodesics in the unit sphere

$$\boxed{\text{As surface of revolution: } (\sqrt{1-u^2}\cos v, \sqrt{1-u^2}\sin v, u)}$$

1. (2nd)Geodesic equation

$$(1-u^2)\dot{v} = f(u)^2\dot{v} = c_0;$$

2. Constant speed (assume arc length parametrization)

$$1 = f'(u)^2\dot{u}^2 + f(u)^2\dot{v}^2 + \dot{u}^2 \Rightarrow 1 - c_0^2 = u^2 + \dot{u}^2.$$

3. Calculate

$$\gamma \times \dot{\gamma} = \left(\frac{\dot{u}\sin v - c_0 u\cos v}{\sqrt{1-u^2}}, \frac{\dot{u}\cos v + c_0 u\sin v}{\sqrt{1-u^2}}, c_0\right)$$

4. $1 - c_0^2 = u^2 + \dot{u}^2 \Rightarrow (\ddot{u} + u)\dot{u} = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}s}(\gamma \times \dot{\gamma}) = 0.$
5. γ lies in a plane passing the origin $\Rightarrow$ big circle.

Looking Back and Forward

Summary

- **Definitions.**
  1. Geodesics. $\nabla_\gamma \dot{\gamma} = 0$.
- **Properties.**
  1. Geodesics must have constant speed.
- **Equations.**
  1. Geodesic equations.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{E}\dot{u} + \mathbb{F}\dot{v}) &= \frac{1}{2}(\mathbb{E}_u \dot{u}^2 + 2\mathbb{F}_u \dot{u}\dot{v} + \mathbb{G}_u \dot{v}^2), \quad (12) \\ \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{F}\dot{u} + \mathbb{G}\dot{v}) &= \frac{1}{2}(\mathbb{E}_v \dot{u}^2 + 2\mathbb{F}_v \dot{u}\dot{v} + \mathbb{G}_v \dot{v}^2). \quad (13) \end{aligned}$$

  2. Alternative formulation of Geodesic equations.

$$\begin{aligned} \ddot{u} + \Gamma^1_{11}\dot{u}^2 + 2\Gamma^1_{12}\dot{u}\dot{v} + \Gamma^1_{22}\dot{v}^2 &= 0 \quad (14) \\ \ddot{v} + \Gamma^2_{11}\dot{u}^2 + 2\Gamma^2_{12}\dot{u}\dot{v} + \Gamma^2_{22}\dot{v}^2 &= 0. \quad (15) \end{aligned}$$

See you next Tuesday!

$\boxed{\text{Review for Midterm II.}}$

- Covers material after the first midterm;
- Similar format as Midterm I.